

LIMITING ABSORPTION PRINCIPLE AND PERFECTLY MATCHED LAYER METHOD FOR DIRICHLET LAPLACIANS IN QUASI-CYLINDRICAL DOMAINS.*

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Abstract. We establish a limiting absorption principle for Dirichlet Laplacians in quasi-cylindrical domains. Outside a bounded set these domains can be transformed onto a semi-cylinder by suitable diffeomorphisms. Dirichlet Laplacians model quantum or acoustically-soft waveguides associated with quasi-cylindrical domains. We construct a uniquely solvable problem with perfectly matched layers of finite length. We prove that solutions of the latter problem approximate outgoing or incoming solutions with an error that exponentially tends to zero as the length of layers tends to infinity. Outgoing and incoming solutions are characterized by means of the limiting absorption principle.

Key words. Perfectly Matched Layers, PML, quasi-cylindrical domains, Dirichlet Laplacian, limiting absorption principle, resonances, compound expansions

AMS subject classifications. 35J25, 65N12, 35Q40, 35P25

1. Introduction. The perfectly matched layer (PML) method, originally introduced in [1], is in common use for the numerical analysis of a wide class of problems. For some of them stability and convergence of the method have been proved mathematically; see, e.g., [2, 3, 4, 5, 6, 7, 20]. In the present paper we develop the PML method for Dirichlet Laplacians in quasi-cylindrical domains $\mathcal{G} \subset \mathbb{R}^{n+1}$, see Fig. 1.1. These are unbounded domains that outside a bounded set can be transformed onto a semi-cylinder by suitable diffeomorphisms. Intuitively, one can understand \mathcal{G} as

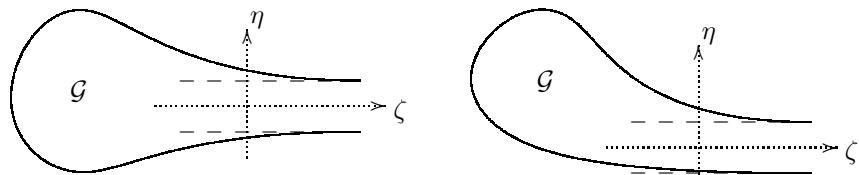


FIG. 1.1. Examples of quasi-cylindrical domains in \mathbb{R}^2 .

a domain whose boundary asymptotically approaches at infinity the boundary of a semi-cylinder $(0, \infty) \times \Omega$, where the cross-section Ω of \mathcal{G} at infinity is a bounded domain in \mathbb{R}^n . Dirichlet Laplacians Δ model quantum or acoustically-soft waveguides associated with quasi-cylindrical domains. In order to characterize outgoing and incoming solutions of the Helmholtz equation $(\Delta - \mu)u = f$ we establish a limiting absorption principle. Then we construct a uniquely solvable problem with PMLs of finite length. This is a Dirichlet problem in the domain \mathcal{G} truncated at a finite distance. We prove that solutions of the latter problem locally approximate outgoing or incoming solutions of the Helmholtz equation with an error that exponentially tends to zero as the length of PMLs tends to infinity. In other words, we prove stability and exponential convergence of the PML method. We find that the rate of exponential

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convergence depends only on the spectral parameter μ and on the infinitely distant cross-section Ω . Thus the rate is the same as in the particular case of a domain \mathcal{G} that coincides with the semi-cylinder $(0, \infty) \times \Omega$ outside a bounded set.

As is known, construction of PMLs is closely related to the complex scaling. The complex scaling involves complex dilation of variables and has a long tradition in mathematical physics and numerical analysis; for a historical account see e.g. [16, 10, 34]. Although there are several papers utilizing different approaches to the complex scaling in waveguide-type of geometry, e.g. [8, 11, 12, 20], the complex scaling has not been used in quasi-cylindrical domains before. Our approach to the complex scaling originates from the one developed in [18] for a Schrödinger operator in \mathbb{R}^3 (see also [16]). Deformations of the Dirichlet Laplacian by means of the complex scaling give rise to an analytic family of non-selfadjoint operators in \mathcal{G} . These operators correspond to a Dirichlet problem with infinite PMLs. Localization of the essential spectra of these operators together with certain relations between their resolvents justifies a limiting absorption principle. For locating the essential spectra we employ methods of the theory of elliptic boundary value problems [24, 25, 28]. Relations between the resolvents are obtained with the help of Hardy spaces of analytic functions. Note that Hardy spaces in context of the complex scaling were originally used in [37, 38]. As we mention in Remark 6.2, our methods also make it possible to develop an analog of the celebrated Aguilar-Balslev-Combes-Simon theory of resonances.

As is typically the case, solutions to the Helmholtz equation satisfying the limiting absorption principle locally coincide with solutions to the problem with infinite PMLs. Moreover, under certain assumptions on the right hand side solutions to the latter problem are of some exponential decay at infinity. This allows us to prove unique solvability of the problem with finite PMLs and establish exponential convergence of the PML method by using the compound expansion technique [27, 25]. In [20] we used a similar approach to study the PML method for inhomogeneous media. Here we study the PML method for a wide class of quasi-cylindrical domains.

This paper is organized as follows. Section 2 consists of preliminaries, where we introduce notations, formulate our assumptions on the quasi-cylindrical domains, and give a formal definition of operators corresponding to the problem with infinite PMLs. In Section 3 we demonstrate that the operators are well-defined and derive some estimates on their coefficients. As shown in Section 4, these operators give rise to an analytic family of m -sectorial operators. In Section 5 we introduce and study Hardy spaces of analytic functions. In Section 6 we formulate and prove a limiting absorption principle. In Section 7 we show that under certain assumptions on the right hand side solutions to the problem with infinite PMLs are of some exponential decay at infinity. Finally, in Section 9 we formulate and study the problem with finite PMLs and prove exponential convergence of the PML method.

2. Preliminaries. In this section we introduce basic notations that are in use throughout the paper. We formulate our assumptions on the quasi-cylindrical domains and introduce differential operators corresponding to the problem with infinite PMLs. Recall that PMLs are artificial strongly absorbing layers designed so that waves coming from a non-PML medium to PMLs do not reflect at the interface.

Let (x, y) and (ζ, η) be two systems of the Cartesian coordinates in \mathbb{R}^{n+1} , $n \geq 1$, such that $x, \zeta \in \mathbb{R}$, while $y = (y_1, \dots, y_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ are in \mathbb{R}^n . Let $\partial_x = \frac{d}{dx}$, $\partial_{y_m} = \frac{d}{dy_m}$, and $\partial_\zeta = \frac{d}{d\zeta}$, $\partial_{\eta_m} = \frac{d}{d\eta_m}$.

Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and the semi-cylinder $\mathbb{R}_+ \times \overline{\Omega}$, where $\mathbb{R}_+ =$

$\{x \in \mathbb{R} : x > 0\}$. We say that $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a quasi-cylinder, if there exists a diffeomorphism

$$\mathbb{R}_+ \times \overline{\Omega} \ni (x, y) \mapsto \varkappa(x, y) = (\zeta, \eta) \in \overline{\mathcal{C}}, \quad (2.1)$$

such that the elements $\varkappa'_{\ell m}(x, \cdot)$ of its Jacobian matrix \varkappa' tend to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\overline{\Omega})$ as $x \rightarrow +\infty$.

Let \mathcal{G} be a domain in \mathbb{R}^{n+1} with smooth boundary $\partial\mathcal{G}$. We suppose that the set $\{(\zeta, \eta) \in \mathcal{G} : \zeta \leq 0\}$ is bounded, and the set $\{(\zeta, \eta) \in \mathcal{G} : \zeta > 0\}$ coincides with a quasi-cylinder \mathcal{C} . (Extension of our results to the case of a domain \mathcal{G} that coincides outside a bounded set with several quasi-cylinders is straightforward.) Following [25], we say that \mathcal{G} is a quasi-cylindrical domain.

Introduce the notation $\nabla_{\zeta\eta} = (\partial_\zeta, \partial_{\eta_1}, \dots, \partial_{\eta_n})^\top$. In the domain \mathcal{G} we consider the Dirichlet Laplacian $\Delta = -\nabla_{\zeta\eta} \cdot \nabla_{\zeta\eta}$, which is initially defined on the set $C_0^\infty(\overline{\mathcal{G}})$ of all smooth compactly supported functions u in $\overline{\mathcal{G}}$ satisfying the Dirichlet boundary condition $u|_{\partial\mathcal{G}} = 0$.

Consider the complex scaling $x \mapsto x + \lambda s(x - r)$ with parameters $r > 0$ and $\lambda \in \mathbb{C}$. Here $s(x)$ is a smooth scaling function possessing the properties:

$$s(x) = 0 \text{ for } x \leq 0, \quad (2.2)$$

$$0 \leq s'(x) \leq 1 \text{ for all } x \in \mathbb{R}, \quad (2.3)$$

$$s'(x) = 1 \text{ for } x \geq C > 0, \quad (2.4)$$

where $s'(x) = \partial_x s(x)$, and C is arbitrary. For all real $\lambda \in (-1, 1)$ the function $\mathbb{R}_+ \ni x \mapsto x + \lambda s(x - r)$ is invertible, and

$$\mathbb{R}_+ \times \overline{\Omega} \ni (x, y) \mapsto \kappa_{\lambda, r}(x, y) = (x + \lambda s(x - r), y) \in \mathbb{R}_+ \times \overline{\Omega}$$

is a selfdiffeomorphism of the semi-cylinder. Therefore

$$\vartheta_{\lambda, r}(\zeta, \eta) = \begin{cases} \varkappa \circ \kappa_{\lambda, r} \circ \varkappa^{-1}(\zeta, \eta) & \text{for } (\zeta, \eta) \in \overline{\mathcal{C}}, \\ (\zeta, \eta) & \text{for } (\zeta, \eta) \in \overline{\mathcal{G}} \setminus \overline{\mathcal{C}}, \end{cases} \quad (2.5)$$

is a selfdiffeomorphism of $\overline{\mathcal{G}}$. In other words, $\vartheta_{\lambda, r}$ with $\lambda \in (-1, 1)$ and $r > 0$ is a scaling of the quasi-cylinder \mathcal{C} along the curvilinear coordinate x . Let $(\vartheta'_{\lambda, r})^\top$ be the transpose of the Jacobian matrix $\vartheta'_{\lambda, r}$. Then $\mathbf{e}_{\lambda, r} = (\vartheta'_{\lambda, r})^\top \vartheta'_{\lambda, r}$ is the matrix coordinate representation of a metric $\mathbf{e}_{\lambda, r}$ on $\overline{\mathcal{G}}$, and

$$\Delta_{\lambda, r} = -(\det \mathbf{e}_{\lambda, r})^{-1/2} \nabla_{\zeta\eta} \cdot (\det \mathbf{e}_{\lambda, r})^{1/2} \mathbf{e}_{\lambda, r}^{-1} \nabla_{\zeta\eta} \quad (2.6)$$

is the Laplace-Beltrami operator on the Riemannian manifold $(\overline{\mathcal{G}}, \mathbf{e}_{\lambda, r})$. As the parameter r increases, the equalities $\vartheta_{\lambda, r}(\zeta, \eta) = (\zeta, \eta)$, $\mathbf{e}_{\lambda, r}(\zeta, \eta) = \text{Id}$, where Id is the $(n+1) \times (n+1)$ identity matrix, and the equality $\Delta_{\lambda, r} = \Delta$ become valid on a larger and larger subset of \mathcal{G} . In the case $\lambda = 0$ the scaling is not applied. Therefore $\mathbf{e}_{0, r} \equiv \mathbf{e}$ is the Euclidean metric and $\Delta_{0, r} \equiv \Delta$.

In order to consider complex values of the scaling parameter λ , we impose additional assumptions on the diffeomorphism (2.1):

- i. the function $\mathbb{R}_+ \ni x \mapsto \varkappa(x, \cdot) \in C^\infty(\overline{\Omega})$ has an analytic continuation from \mathbb{R}_+ to some sector

$$\mathbb{S}_\alpha = \{z \in \mathbb{C} : |\arg z| < \alpha < \pi/4\}; \quad (2.7)$$

ii. the elements $\varkappa'_{\ell m}(z, \cdot)$ of the Jacobian matrix \varkappa' tend to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\overline{\Omega})$ uniformly in $z \in \mathbb{S}_\alpha$ as $z \rightarrow \infty$.

For instance, the assumptions *i,ii* are satisfied for the following quasi-cylinders:

$$\mathcal{C} = \{(\zeta, \eta) \in \mathbb{R}^2 : (\zeta, \eta) = (x, y + \log(x + 2)), x \in \mathbb{R}_+, y \in [0, 1]\},$$

$$\mathcal{C} = \left\{ (\zeta, \eta) \in \mathbb{R}^2 : \zeta = \int_0^x \varphi(t) dt, \eta = y\psi(x), x \in \mathbb{R}_+, y \in [0, 1] \right\},$$

where as $\varphi(x)$ and $\psi(x)$ we can take the functions $1, 1 + e^{-x}, 1 + (x + 1)^{-s}$ with $s > 0$, $1 + 1/\log(x + 2)$, $1 + 1/\log(1 + \log(x + 2))$, and so on. These examples show that quasi-cylinders can have very different shapes comparing with the semi-cylinder.

In the next section we will show that for all sufficiently large $r > 0$ the assumptions *i, ii* on \varkappa together with (2.2) and (2.3) lead to the analyticity of the coefficients of the differential operator (2.6) with respect to the scaling parameter λ in the disk

$$\mathcal{D}_\alpha = \{\lambda \in \mathbb{C} : |\lambda| < \sin \alpha < 1/\sqrt{2}\}. \quad (2.8)$$

Thus the equality (2.6) defines $\Delta_{\lambda, r}$ for all $\lambda \in \mathcal{D}_\alpha$. Clearly, $\Delta_{\lambda, r}$ coincides with the Dirichlet Laplacian Δ on the set

$$\mathcal{G}_r = (\mathcal{G} \setminus \mathcal{C}) \cup \{(\zeta, \eta) \in \mathcal{C} : (\zeta, \eta) = \varkappa(x, y), x < r, y \in \Omega\}. \quad (2.9)$$

We will show that on the set $\mathcal{G} \setminus \mathcal{G}_r$ the operator $\Delta_{\lambda, r}$ with $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$ describes an infinite PML for Δ . In the case $\Im \lambda > 0$ (resp. $\Im \lambda < 0$) this PML is an artificial nonreflective strongly absorbing layer for the outgoing (resp. incoming) solutions.

REMARK 2.1. *For simplicity we consider in this paper only Dirichlet Laplacians. However, similar methods can be used to develop and study PML method for the Schrödinger operator $\Delta + V$ in \mathcal{G} , where Δ is the Dirichlet Laplacian and $V \in C^\infty(\overline{\mathcal{G}})$ is a real-valued potential with the following properties: for some $r_0 > 0$ and $\alpha > 0$ the function $x \mapsto V \circ \varkappa(x, \cdot) \in L^2(\Omega)$ extends by analyticity to the sector $\{z \in \mathbb{C} : |\arg(z - r_0)| < \alpha\}$, where for all $y \in \overline{\Omega}$ we have $|V \circ \varkappa(z, y)| \leq C(|z|) \rightarrow 0$ as $z \rightarrow \infty$. One can also include into consideration potentials with moderate local singularities and relatively bounded operator-valued potentials.*

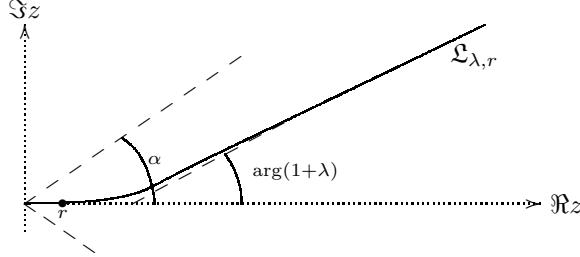
3. Construction of infinite PMLs. In this section we show that for all sufficiently large $r > 0$ the differential operator (2.6) is well-defined for complex values of the scaling parameter λ in the disk (2.8). We also obtain some estimates on the matrix $\mathbf{e}_{\lambda, r}$.

Consider the quasi-cylinder $\overline{\mathcal{C}}$ as a manifold endowed with the Euclidean metric \mathbf{e} . We will use the coordinates (ζ, η) in $\overline{\mathcal{G}}$ and (x, y) in $\mathbb{R}_+ \times \overline{\Omega}$, and identify the Riemannian metrics on $\overline{\mathcal{G}}$ and $\mathbb{R}_+ \times \overline{\Omega}$ with their matrix coordinate representations. Let $\mathbf{g} = \varkappa^* \mathbf{e}$ be the pullback of the metric \mathbf{e} by the diffeomorphism \varkappa in (2.1). Then the matrix $\mathbf{g} = [\mathbf{g}_{\ell m}]_{\ell, m=1}^{n+1}$ is given by the equality $\mathbf{g} = (\varkappa')^\top \varkappa'$, where $(\varkappa')^\top$ is the transpose of the Jacobian \varkappa' . Since the diffeomorphism \varkappa satisfies the assumptions *i,ii* of Section 2, we conclude that the metric matrix elements

$$\mathbb{S}_\alpha \ni z \mapsto \mathbf{g}_{\ell m}(z, \cdot) \in C^\infty(\overline{\Omega}) \quad (3.1)$$

are analytic functions. Moreover, $\mathbf{g}_{\ell m}(z, \cdot)$ tends to the Kronecker delta $\delta_{\ell m}$ in the space $C^\infty(\overline{\Omega})$ uniformly in $z \in \mathbb{S}_\alpha$ as $z \rightarrow \infty$ or, equivalently, we have

$$|\partial_y^q(\mathbf{g}_{\ell m}(z, y) - \delta_{\ell m})| \leq C_q(|z|) \rightarrow 0 \text{ as } z \rightarrow \infty, z \in \mathbb{S}_\alpha, y \in \overline{\Omega}, |q| \geq 0; \quad (3.2)$$

FIG. 3.1. The curve $\mathfrak{L}_{\lambda,r}$ for complex values of λ .

here $\partial_y^q = \partial_{y_1}^{q_1} \partial_{y_2}^{q_2} \dots \partial_{y_n}^{q_n}$ with a multiindex $q = (q_1, \dots, q_n)$, and $|q| = \sum q_j$.

Consider the selfdiffeomorphism $\kappa_{\lambda,r}$, $\lambda \in (-1, 1)$, of the semi-cylinder $\mathbb{R}_+ \times \overline{\Omega}$. We define the metric $g_{\lambda,r} = \kappa_{\lambda,r}^* g$ on $\mathbb{R}_+ \times \overline{\Omega}$ as the pullback of the metric g by $\kappa_{\lambda,r}$. As a result we get the manifold $(\mathbb{R}_+ \times \overline{\Omega}, g_{\lambda,r})$ parameterized by $\lambda \in (-1, 1)$ and $r > 0$. We deduce

$$g_{\lambda,r}(x, y) = \text{diag} \{1 + \lambda s'(x - r), \text{Id}\} g(x + \lambda s(x - r), y) \text{diag} \{1 + \lambda s'(x - r), \text{Id}\}, \quad (3.3)$$

where Id stands for the $n \times n$ -identity matrix, and $\text{diag} \{1 + \lambda s'(x - r), \text{Id}\}$ is the Jacobian of $\kappa_{\lambda,r}$.

Let us consider complex values of the scaling parameter λ . We suppose that λ is in the complex disk \mathcal{D}_α , where α is the same as in our assumptions on the partial analytic regularity of the diffeomorphism \varkappa ; cf. (2.7), (2.8). The curve $\mathfrak{L}_{\lambda,r} = \{z \in \mathbb{C} : z = x + \lambda s(x - r), x > 0\}$ lies in the sector \mathbb{S}_α , see Fig. 3.1. We define the matrix $g_{\lambda,r}$ for all non-real λ in the disk by the equality (3.3), where $g(x + \lambda s(x - r), y)$ stands for the value of the analytic in $z \in \mathbb{S}_\alpha$ function $g(z, y)$ at $z = x + \lambda s(x - r)$. By analyticity in λ we conclude that $g_{\lambda,r}$ is a complex symmetric matrix, the Schwarz reflection principle gives $\overline{g_{\lambda,r}} = g_{\overline{\lambda},r}$, where the overline stands for the complex conjugation. If $\lambda \in \mathcal{D}_\alpha$ is non-real, then the matrix $g_{\lambda,r}$ does not correspond to a Riemannian metric. However, $g_{\lambda,r}$ is invertible for all $\lambda \in \mathcal{D}_\alpha$ provided $r > 0$ is sufficiently large. Indeed, $s(x - r) = 0$ and $g_{\lambda,r}^{-1}(x, y) = g^{-1}(x, y)$ for all $x < r$. On the other hand, the matrix $g(x + \lambda s(x - r), y)$ from (3.3) is invertible for $x \geq r$ with large $r > 0$ as it is only little different from the identity matrix; the latter fact is a consequence of (3.2) and the inequality $|x + \lambda s(x - r)| \geq r$.

The derivatives $\partial_x^p \partial_y^q g_{\lambda,r}^{-1}$ are analytic functions of $\lambda \in \mathcal{D}_\alpha$. From the conditions (3.1) and (3.2) together with (2.4) and (3.3) it follows that

$$\left\| \partial_x^p \partial_y^q (g_{\lambda,r}^{-1}(x, y) - \text{diag}\{(1 + \lambda)^{-2}, \text{Id}\}) \right\| \rightarrow 0 \text{ as } x \rightarrow +\infty, y \in \overline{\Omega}, \lambda \in \mathcal{D}_\alpha, \quad (3.4)$$

and the estimate

$$\left\| \partial_x^p \partial_y^q (g_{\lambda,r}^{-1}(x, y) - \text{diag}\{(1 + \lambda s'(x - r))^{-2}, \text{Id}\}) \right\| \leq C_{pq}(r) \quad (3.5)$$

holds uniformly in $(x, y) \in [r, \infty) \times \overline{\Omega}$ and $\lambda \in \mathcal{D}_\alpha$, where $\|\cdot\|$ is the matrix norm $\|A\| = \max_{\ell m} |a_{\ell m}|$, and $p + |q| \geq 0$. The constants $C_{pq}(r)$ in (3.5) tend to zero as $r \rightarrow +\infty$.

We define the complex scaling $\vartheta_{\lambda,r}$ for all λ in the disk \mathcal{D}_α by the equality (2.5), where $\varkappa \circ \kappa_{\lambda,r}(x, y)$ is the value of the analytic in $z \in \mathbb{S}_\alpha$ function $\varkappa(z, y)$ at the point

$z = x + \lambda s(x - r)$. Consider the matrix $\mathbf{e}_{\lambda,r} = (\vartheta'_{\lambda,r})^\top \vartheta'_{\lambda,r}$. It is clear that for all $(\zeta, \eta) \in \overline{\mathcal{G}} \setminus \overline{\mathcal{C}}$ the matrix $\mathbf{e}_{\lambda,r}(\zeta, \eta)$ coincides with the $(n+1) \times (n+1)$ -identity. For all real $\lambda \in \mathcal{D}_\alpha$ we have

$$\mathbf{g}_{\lambda,r}(x, y) = (\varkappa'(x, y))^\top (\mathbf{e}_{\lambda,r} \circ \varkappa(x, y)) \varkappa'(x, y), \quad (x, y) \in \mathbb{R}_+ \times \overline{\Omega}, \quad (3.6)$$

where $\mathbf{g}_{\lambda,r} = \varkappa^* \mathbf{e}_{\lambda,r}$ is the pullback of the corresponding metric $\mathbf{e}_{\lambda,r}$ on \mathcal{C} by the diffeomorphism \varkappa . Therefore $\mathbf{e}_{\lambda,r}$ is analytic in $\lambda \in \mathcal{D}_\alpha$ and invertible for all sufficiently large $r > 0$. By analyticity in λ we conclude that $\mathbf{e}_{\lambda,r}(\zeta, \eta)$ is a complex symmetric matrix, the Schwarz reflection principle gives $\overline{\mathbf{e}_{\lambda,r}} = \mathbf{e}_{\overline{\lambda},r}$.

Differentiating the equality (3.6), we see that the matrices $\partial_\zeta^p \partial_\eta^q \mathbf{e}_{\lambda,r}$ and $\partial_\zeta^p \partial_\eta^q \mathbf{e}_{\lambda,r}^{-1}$ are analytic in $\lambda \in \mathcal{D}_\alpha$. Moreover, from (3.4), (3.5), and our assumptions on \varkappa we obtain

$$\begin{aligned} \left\| \partial_\zeta^p \partial_\eta^q (\mathbf{e}_{\lambda,r}^{-1}(\zeta, \eta) - \text{diag}\{(1 + \lambda)^{-2}, \text{Id}\}) \right\| &\rightarrow 0 \text{ as } \zeta \rightarrow +\infty, \\ \left\| \partial_\zeta^p \partial_\eta^q (\mathbf{e}_{\lambda,r}^{-1}(\zeta, \eta) - \text{diag}\{(1 + \lambda s'_r(\zeta, \eta))^{-2}, \text{Id}\}) \right\| &\leq c(r), \end{aligned} \quad (3.7)$$

where $p + |q| \leq 1$, and $c(r) \rightarrow 0$ as $r \rightarrow +\infty$.

Here $(\zeta, \eta) \in \overline{\mathcal{C}}$ and $s'_r(\zeta, \eta)$ stands for the function $s'(x - r)$ written in the coordinates (ζ, η) . We extend s'_r from $\overline{\mathcal{C}}$ to $\overline{\mathcal{G}}$ by zero. Note that the estimate (3.7) remains valid for all $(\zeta, \eta) \in \overline{\mathcal{G}}$ and the constant $c(r)$ is independent of $\lambda \in \mathcal{D}_\alpha$ and $(\zeta, \eta) \in \overline{\mathcal{G}}$. Now we see that for all sufficiently large $r > 0$ the differential operator (2.6) is well-defined for all λ in the disk \mathcal{D}_α , and its coefficients are subjected to the estimate (3.7).

4. Analytic families of operators. In this section we study the unbounded operator $\Delta_{\lambda,r}$ in the Hilbert space $L^2(\mathcal{G})$ with the usual norm

$$\|u; L^2(\mathcal{G})\| = \left(\int_{\mathcal{G}} |u(\zeta, \eta)|^2 d\zeta d\eta \right)^{1/2}.$$

The operator $\Delta_{\lambda,r}$ is initially defined on the set $C_0^\infty(\overline{\mathcal{G}})$. We show that the operator is closable, and the closure defines an analytic family $\mathcal{D}_\alpha \ni \lambda \mapsto \Delta_{\lambda,r}$ of type (B) [23]. In a standard way this implies that the resolvent $(\Delta_{\lambda,r} - \mu)^{-1}$ is an analytic function of λ and μ on some open subset of $\mathcal{D}_\alpha \times \mathbb{C}$. The latter fact will be used for justification of a limiting absorption principle in Section 6.

We intend to show that the operator $\Delta_{\lambda,r}$ in $L^2(\mathcal{G})$ with the domain $C_0^\infty(\overline{\mathcal{G}})$ is sectorial and to study its Friedrichs extension. With the operator $\Delta_{\lambda,r}$ we associate the quadratic form

$$q_{\lambda,r}[u, u] = \int_{\mathcal{G}} \langle (\det \mathbf{e}_{\lambda,r})^{1/2} \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} u \rangle d\zeta d\eta, \quad (4.1)$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product in \mathbb{C}^{n+1} and $u \in C_0^\infty(\overline{\mathcal{G}})$. Let $\langle \cdot, \cdot \rangle$ stand for the inner product in $L^2(\mathcal{G})$. We represent the quadratic form as follows:

$$\begin{aligned} q_{\lambda,r}[u, u] &= (-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u) \\ &+ \int_{\mathcal{G}} \langle (\det \mathbf{e}_{\lambda,r})^{1/2} \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} \rangle d\zeta d\eta. \end{aligned} \quad (4.2)$$

For the first term in the right hand side of (4.2) we prove the following lemma.

LEMMA 4.1. *Assume that $r > 0$ is sufficiently large. Then there exist $\varphi < \pi/2$ and $\delta > 0$ such that for all $\lambda \in \mathcal{D}_\alpha$ and $u \in C_0^\infty(\overline{\mathcal{G}})$ we have*

$$|\arg(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u)| \leq \varphi, \quad \delta(\Delta u, u) \leq \Re(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u) \leq \delta^{-1}(\Delta u, u).$$

In other words, the form $(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u)$ is sectorial.

Proof. It is clear that

$$(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u) = \int_{\mathcal{G}} \langle \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u \rangle d\zeta d\eta. \quad (4.3)$$

Let us estimate the numerical range of the matrix $\mathbf{e}_{\lambda,r}^{-1}(\zeta, \eta)$. We shall rely on the estimate (3.7). Let $\xi = \nabla_{\zeta\eta} u(\zeta, \eta) \in \mathbb{C}^{n+1}$. Observe that by virtue of $0 \leq s'_r(\zeta, \eta) \leq 1$ and $|\lambda| < \sin \alpha < 2^{-1/2}$ we have

$$\begin{aligned} |\xi|^2/4 &\leq \left| \overline{\xi} \cdot \text{diag}\{(1 + \lambda s'_r(\zeta, \eta))^{-2}, \text{Id}\} \xi \right| \leq 12|\xi|^2, \\ |\arg(\overline{\xi} \cdot \text{diag}\{(1 + \lambda s'_r(\zeta, \eta))^{-2}, \text{Id}\} \xi)| &< 2\alpha. \end{aligned} \quad (4.4)$$

Since r is sufficiently large, the constant $c(r)$ in (3.7) is small. In particular $c(r)$ meets the estimate $4(n+1)^2 c(r) \leq \sin(\sigma/2)$ with some $\sigma \in (0, \pi/2 - 2\alpha)$. Then (3.7) together with (4.4) gives

$$\left| \arg(\overline{\xi} \cdot \mathbf{e}_{\lambda,r}^{-1}(\zeta, \eta) \xi) \right| \leq \varphi < \pi/2, \quad \delta|\xi|^2 \leq \left| \overline{\xi} \cdot \mathbf{e}_{\lambda,r}^{-1}(\zeta, \eta) \xi \right| \leq \delta^{-1}|\xi|^2, \quad (4.5)$$

uniformly in $\lambda \in \mathcal{D}_\alpha$ and $(\zeta, \eta) \in \overline{\mathcal{G}}$, where $\varphi = 2\alpha + \sigma$ and

$$\delta = \min \left\{ 1/4 - (n+1)^2 c(r), (12 + (n+1)^2 c(r))^{-1} \right\}.$$

Taking into account (4.3) we complete the proof. \square

REMARK 4.2. *Throughout the paper we say that $r > 0$ is sufficiently large if the matrix $\mathbf{e}_{\lambda,r}(\zeta, \eta)$ is invertible and its inverse meets the estimates (4.5) uniformly in $(\zeta, \eta) \in \overline{\mathcal{G}}$ and $\lambda \in \mathcal{D}_\alpha$.*

In the next lemma we show that in the right hand side of (4.2) the second term has an arbitrarily small relative bound with respect to the first term uniformly in $\lambda \in \mathcal{D}_\alpha$.

LEMMA 4.3. *For any $\epsilon > 0$ and $u \in C_0^\infty(\overline{\mathcal{G}})$ the estimate*

$$\begin{aligned} &\left| \int_{\mathcal{G}} \langle (\det \mathbf{e}_{\lambda,r})^{1/2} \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} \rangle d\zeta d\eta \right| \\ &\leq \epsilon |(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u)| + C\epsilon^{-1} \|u; L^2(\mathcal{G})\|^2 \end{aligned}$$

holds, where the constant C is independent of ϵ , u , and $\lambda \in \mathcal{D}_\alpha$.

Proof. We have

$$\begin{aligned} &\left| \int_{\mathcal{G}} \langle (\det \mathbf{e}_{\lambda,r})^{1/2} \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} \rangle d\zeta d\eta \right| \\ &\leq C \left(\int_{\mathcal{G}} |\langle \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} \rangle|^2 d\zeta d\eta \right)^{1/2} \left(\int_{\mathcal{G}} |u|^2 d\zeta d\eta \right)^{1/2} \\ &\leq C \left(\tilde{\epsilon} \int_{\mathcal{G}} |\langle \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} \rangle|^2 d\zeta d\eta + \tilde{\epsilon}^{-1} \|u; L^2(\mathcal{G})\|^2 \right) \end{aligned} \quad (4.6)$$

with arbitrarily small $\tilde{\epsilon} > 0$ and $C = \sup_{(\zeta, \eta)} (\det \mathbf{e}_{\lambda, r}(\zeta, \eta))^{1/2} < \infty$, cf. (3.7).

From (4.5) it follows that

$$|\Im\{\bar{\xi} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\xi\}| \leq (\tan \varphi) \Re\{\bar{\xi} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\xi\},$$

where the form $\Re\{\bar{\xi} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\xi\}$ defines an inner product in \mathbb{C}^{n+1} . This and the Cauchy-Schwarz inequality give

$$|\bar{\tau} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\xi|^2 \leq (1 + \tan \varphi)^2 \Re\{\bar{\xi} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\xi\} \Re\{\bar{\tau} \cdot \mathbf{e}_{\lambda, r}^{-1}(\zeta, \eta)\tau\}, \quad (4.7)$$

cf. [23, Chapter VI.2]. We substitute $\xi = \nabla_{\zeta\eta} u(\zeta, \eta)$ and $\tau = \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda, r}(\zeta, \eta))^{-1/2}$. Thanks to (3.7) we have the uniform bound

$$|\nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda, r}(\zeta, \eta))^{-1/2}|^2 \leq c, \quad \lambda \in \mathcal{D}_\alpha, (\zeta, \eta) \in \bar{\mathcal{G}}.$$

This together with (4.7) and (4.5) implies

$$|\langle \mathbf{e}_{\lambda, r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda, r})^{-1/2} \rangle|^2 \leq c\delta^{-1}(1 + \tan \varphi)^2 \Re\langle \mathbf{e}_{\lambda, r}^{-1} \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u \rangle.$$

Now we make use of (4.6) and establish the assertion for $C = C^2 c\delta^{-1}(1 + \tan \varphi)^2$ and an arbitrarily small $\epsilon = Cc\delta^{-1}(1 + \tan \varphi)^2\tilde{\epsilon}$. \square

As a consequence of the equality (4.2) and Lemmas 4.1, 4.3 for all sufficiently large $r > 0$ we obtain

$$|\arg(q_{\lambda, r}[u, u] + \gamma\|u; L^2(\mathcal{G})\|^2)| \leq \varphi < \pi/2, \quad (4.8)$$

$$\delta q_{0, r}[u, u] - \gamma\|u; L^2(\mathcal{G})\|^2 \leq \Re q_{\lambda, r}[u, u] \leq \delta^{-1}(q_{0, r}[u, u] + \|u; L^2(\mathcal{G})\|^2) \quad (4.9)$$

with some angle φ and some positive constants δ and γ , which are independent of $u \in C_0^\infty(\bar{\mathcal{G}})$ and $\lambda \in \mathcal{D}_\alpha$. The symmetric form $q_{0, r}[u, u] = \int_{\mathcal{G}} \langle \nabla_{\zeta\eta} u, \nabla_{\zeta\eta} u \rangle d\zeta d\eta$ is independent of r , it corresponds to the Dirichlet Laplacian $\Delta \equiv \Delta_{0, r}$. Clearly, $q_{\lambda, r}[u, u] = (\Delta_{\lambda, r} u, u)$. Estimate (4.8) implies that the numerical range

$$\{\mu \in \mathbb{C} : \mu = (\Delta_{\lambda, r} u, u), u \in C_0^\infty(\bar{\mathcal{G}}), \|u; L^2(\mathcal{G})\| = 1\}$$

is a subset of the sector $\{\mu \in \mathbb{C} : |\arg(\mu + \gamma)| \leq \varphi < \pi/2\}$. By definition this means that the operator $\Delta_{\lambda, r}$ with the domain $C_0^\infty(\bar{\mathcal{G}})$ is sectorial.

We introduce the Sobolev space $\mathring{H}^1(\mathcal{G})$ as the completion of the set $C_0^\infty(\bar{\mathcal{G}})$ with respect to the norm

$$\|u; \mathring{H}^1(\mathcal{G})\| = \sqrt{q_{0, r}[u, u] + \|u; L^2(\mathcal{G})\|^2}.$$

Recall that *i*) a sequence $\{u_j\}$ is said to be $q_{\lambda, r}$ -convergent, if u_j is in the domain of $q_{\lambda, r}$, $\|u_j - u; L^2(\mathcal{G})\| \rightarrow 0$ and $q_{\lambda, r}[u_j - u_m, u_j - u_m] \rightarrow 0$ as $j, m \rightarrow \infty$; *ii*) the form $q_{\lambda, r}$ is closed, if every $q_{\lambda, r}$ -convergent sequence $\{u_j\}$ has a limit u in the domain of $q_{\lambda, r}$, and $q_{\lambda, r}[u - u_j, u - u_j] \rightarrow 0$. From (4.8), (4.9) it immediately follows that $q_{\lambda, r}$ with the domain $\mathring{H}^1(\mathcal{G})$ is a closed densely defined sectorial form [23, 34], and its sector $\{\mu \in \mathbb{C} : |\arg(\mu + \gamma)| \leq \varphi\}$ is independent of $\lambda \in \mathcal{D}_\alpha$. As known [23, Chapter VI.2.1], to every closed densely defined sectorial form there corresponds a unique m -sectorial operator. Namely, to the form $q_{\lambda, r}$ there corresponds a unique

m-sectorial operator $\Delta_{\lambda,r}$ in $L^2(\mathcal{M})$ such that its sector is the sector of $q_{\lambda,r}$, the domain $D(\Delta_{\lambda,r})$ is dense in $\dot{H}^1(\mathcal{G})$, and $q_{\lambda,r}[u,v] = (\Delta_{\lambda,r}u, v)$ for all $u \in D(\Delta_{\lambda,r})$ and $v \in \dot{H}^1(\mathcal{G})$. (Here and elsewhere m-sectorial means that the numerical range $\{\mu = (Au, u)_{\mathcal{H}} \in \mathbb{C} : u \in D(A), (u, u)_{\mathcal{H}} = 1\}$ and the spectrum $\sigma(A)$ of a closed unbounded operator A in a Hilbert space \mathcal{H} with the inner product $(\cdot, \cdot)_{\mathcal{H}}$ both lie in some sector $\{\mu \in \mathbb{C} : \arg(\mu - \gamma) \leq \varphi\}$ with $\gamma \in \mathbb{R}$ and $\varphi < \pi/2$.) In particular, to the symmetric nonnegative form $q_{0,r}$ there corresponds the selfadjoint Dirichlet Laplacian $\Delta \equiv \Delta_{0,r}$. The m-sectorial operator $\Delta_{\lambda,r}$ in $L^2(\mathcal{M})$ is the Friedrichs extension of the sectorial operator $\Delta_{\lambda,r}$ defined on $C_0^\infty(\overline{\mathcal{G}})$, see [23, Chapter VI.2.3]. As we show in assertion 1 of the next proposition the set $C_0^\infty(\overline{\mathcal{G}})$ is a core of the Friedrichs extension.

PROPOSITION 4.4. *Assume that $r > 0$ is sufficiently large. Then the following assertions are valid.*

1. *For $\lambda \in \mathcal{D}_\alpha$ the m-sectorial operator $\Delta_{\lambda,r}$ in $L^2(\mathcal{G})$ with the domain $D(\Delta_{\lambda,r})$ is the closure of the operator $\Delta_{\lambda,r}$ defined on the set $C_0^\infty(\overline{\mathcal{G}})$.*
2. *The family of m-sectorial operators $\mathcal{D}_\alpha \ni \lambda \mapsto \Delta_{\lambda,r}$ in $L^2(\mathcal{G})$ is an analytic family of type (B).*
3. *The resolvent $\Gamma \ni (\lambda, \mu) \mapsto (\Delta_{\lambda,r} - \mu)^{-1} : L^2(\mathcal{G}) \rightarrow L^2(\mathcal{G})$ is an analytic function of two variables on the set $\Gamma = \{(\lambda, \mu) : \lambda \in \mathcal{D}_\alpha, \mu \in \mathbb{C} \setminus \sigma(\Delta_{\lambda,r})\}$, where $\sigma(\Delta_{\lambda,r})$ is the spectrum of $\Delta_{\lambda,r}$.*

Proof. Consider the domain $D(\Delta_{\lambda,r})$ as a Hilbert space endowed with the graph norm $\|u; D(\Delta_{\lambda,r})\| = \|u; L^2(\mathcal{G})\| + \|\Delta_{\lambda,r}u; L^2(\mathcal{G})\|$. Let μ be a point outside of the sector of the m-sectorial operator $\Delta_{\lambda,r}$. Then the set

$$C(\Delta_{\lambda,r}) = \{u : u = (\Delta_{\lambda,r} - \mu)^{-1}f, f \in C_0^\infty(\overline{\mathcal{G}})\}$$

is dense in $D(\Delta_{\lambda,r})$ because the resolvent $(\Delta_{\lambda,r} - \mu)^{-1} : L^2(\mathcal{G}) \rightarrow D(\Delta_{\lambda,r})$ is bounded and the set $C_0^\infty(\overline{\mathcal{G}})$ is dense in $L^2(\mathcal{G})$. From (4.5) it follows the estimate

$$\Re(\xi \cdot e_{\lambda,r}^{-1}(\zeta, \eta)\xi) \geq c|\xi|^2, \quad \xi \in \mathbb{R}^{n+1}, \lambda \in \mathcal{D}_\alpha, (\zeta, \eta) \in \overline{\mathcal{G}}, \quad (4.10)$$

on the principal symbol of $\Delta_{\lambda,r}$, where $c > 0$. Hence $\Delta_{\lambda,r}$ is a strongly elliptic operator. As is well-known, a strongly elliptic operator and the Dirichlet boundary condition set up an elliptic boundary value problem, e.g. [26]. The usual argument on the regularity of solutions to the elliptic boundary value problems [26, 25] implies that the set $C(\Delta_{\lambda,r})$ consists of smooth in $\overline{\mathcal{G}}$ functions u with $u|_{\partial\mathcal{G}} = 0$. Multiplying $u \in C(\Delta_{\lambda,r})$ by appropriate cutoff functions χ_j with expanding compact supports $\text{supp } \chi_j \subset \text{supp } \chi_{j+1}$, it is easy to see that for any $u \in C(\Delta_{\lambda,r})$ there is a sequence $\{\chi_j u\}_{j=1}^\infty$ such that $\chi_j u \in C_0^\infty(\overline{\mathcal{G}})$ tends to u in $D(\Delta_{\lambda,r})$ as $j \rightarrow +\infty$. Assertion 1 is proven.

The family $\mathcal{D}_\alpha \ni \lambda \mapsto q_{\lambda,r}$ is analytic in the sense of Kato (i.e. $q_{\lambda,r}$ is a closed densely defined sectorial form, its domain $\dot{H}^1(\mathcal{G})$ is independent of λ , and the function $\mathcal{D}_\alpha \ni \lambda \mapsto q_{\lambda,r}[u, u]$ is analytic for any $u \in \dot{H}^1(\mathcal{G})$). By definition [23, 34] this means that the family of m-sectorial operators $\mathcal{D}_\alpha \ni \lambda \mapsto \Delta_{\lambda,r}$ is an analytic family of type (B). This proves assertion 2.

As is well-known [23, 34], any analytic family of type (B) is also an analytic family of operators in the sense of Kato. Now a standard argument justifies assertion 3; see, e.g., [34, Theorem XII.7]. \square

5. Spaces of analytic functions. We have shown that for all μ outside the sector of the family $\mathcal{D}_\alpha \ni \lambda \mapsto \Delta_{\lambda,r}$ of m-sectorial operators the resolvent $(\Delta_{\lambda,r} - \mu)^{-1}$

is an analytic function of λ , see Proposition 4.4.3. In order to get some relations between the resolvents $(\Delta - \mu)^{-1}$ and $(\Delta_{\lambda,r} - \mu)^{-1}$ we will use a sufficiently large Hilbert space $\mathcal{H}_\alpha(\mathcal{G})$ of analytic functions

$$\mathcal{D}_\alpha \ni \lambda \mapsto f \circ \vartheta_{\lambda,r} \in L^2(\mathcal{G}); \quad (5.1)$$

here $\vartheta_{\lambda,r}$ is the complex scaling (2.5). The goal of this section is two-fold:

1. To introduce a Hilbert space $\mathcal{H}_\alpha(\mathcal{G})$, which is sufficiently large in the sense that for any $\lambda \in \mathcal{D}_\alpha$ and $r > 0$ the set $\{f \circ \vartheta_{\lambda,r} \in L^2(\mathcal{G}) : f \in \mathcal{H}_\alpha(\mathcal{G})\}$ is dense in $L^2(\mathcal{G})$;
2. To derive the uniform in $\lambda \in \mathcal{D}_\alpha$ and $f \in \mathcal{H}_\alpha(\mathcal{G})$ estimate

$$\|f \circ \vartheta_{\lambda,r}; L^2(\mathcal{G})\| \leq C_r \|f; \mathcal{H}_\alpha(\mathcal{G})\|, \quad r > 0. \quad (5.2)$$

Introduce the Hardy class $\mathfrak{H}(\mathbb{S}_\alpha)$ of all analytic functions $\mathbb{S}_\alpha \ni z \mapsto F(z) \in L^2(\Omega)$ satisfying the uniform in ψ estimate

$$\int_0^\infty \|F(e^{i\psi}x); L^2(\Omega)\|^2 dx \leq C_F, \quad \psi \in (-\alpha, \alpha). \quad (5.3)$$

Below we cite some facts from the theory of Hardy classes.

LEMMA 5.1.

1. Every $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ has boundary limits $F_\pm \in L^2(\mathbb{R}_+ \times \Omega)$ such that

$$\int_0^\infty \|F(e^{i\psi}x) - F_\pm(x); L^2(\Omega)\|^2 dx \rightarrow 0 \text{ as } \psi \rightarrow \pm\alpha.$$

2. The Hardy class $\mathfrak{H}(\mathbb{S}_\alpha)$ endowed with the norm

$$\|F; \mathfrak{H}(\mathbb{S}_\alpha)\| = (\|F_-; L^2(\mathbb{R}_+ \times \Omega)\|^2 + \|F_+; L^2(\mathbb{R}_+ \times \Omega)\|^2)^{1/2}$$

is a Hilbert space.

3. For every compact set $\mathfrak{K} \subset \mathbb{S}_\alpha$ there is an independent of $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ constant $C(\mathfrak{K})$ such that for all $z \in \mathfrak{K}$ we have

$$\|F(z); L^2(\Omega)\| \leq C(\mathfrak{K}) \|F; \mathfrak{H}(\mathbb{S}_\alpha)\|.$$

4. For any $F \in \mathfrak{H}(\mathbb{S}_\alpha)$, $z \in \mathbb{S}_\alpha$, and $\psi \in (-\alpha, \alpha)$ we have

$$\int_0^\infty \|F(z + e^{i\psi}x); L^2(\Omega)\|^2 dx \leq \|F; \mathfrak{H}(\mathbb{S}_\alpha)\|^2.$$

Proof. Assertions 1–3 are direct consequences of well-known facts from the theory of Hardy spaces of functions analytic in strips, e.g. [35]. In fact, the proof reduces to the conformal mapping of the sector \mathbb{S}_α to the strip $\{z \in \mathbb{C} : -\alpha < \Im z < \alpha\}$, we omit the details.

Let us prove assertion 4. A standard argument, see e.g. [36], shows that any $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ can be represented by the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\alpha} F_+(x)}{z - e^{i\alpha}x} dx - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-i\alpha} F_-(x)}{z - e^{-i\alpha}x} dx, \quad z \in \mathbb{S}_\alpha, \quad (5.4)$$

where we assume that the boundary limits $F_{\pm}(x)$ are extended to $x < 0$ by zero. The integrals are absolutely convergent in the space $L^2(\Omega)$. As is well-known [35], the first integral in (5.4) defines an element f_+ of the Hardy space $\mathfrak{H}(\mathbb{C}_\alpha^-)$ of $L^2(\Omega)$ -valued functions analytic in the half-plane $\mathbb{C}_\alpha^- = \{z \in \mathbb{C} : \Im(e^{-i\alpha}z) < 0\}$. As shown in [37, 38], the norm in $\mathfrak{H}(\mathbb{C}_\alpha^-)$ can be defined by the equality

$$\|f_+; \mathfrak{H}(\mathbb{C}_\alpha^-)\| = \left(\sup_{\psi \in (\alpha - \pi, \alpha)} \int_0^\infty \|f_+(z_+ + e^{i\psi}x); L^2(\Omega)\|^2 dx \right)^{1/2}, \quad e^{-i\alpha}z_+ \in \mathbb{R}.$$

Then $\|f_+; \mathfrak{H}(\mathbb{C}_\alpha^-)\| = \|F_+; L^2(\mathbb{R}_+ \times \Omega)\|$. Similarly, the second integral in (5.4) defines a function f_- from the Hardy space $\mathfrak{H}(\mathbb{C}_\alpha^+)$ in $\mathbb{C}_\alpha^+ = \{z \in \mathbb{C} : \Im(e^{i\alpha}z) > 0\}$, and $\|f_-; \mathfrak{H}(\mathbb{C}_\alpha^+)\| = \|F_-; L^2(\mathbb{R}_+ \times \Omega)\|$, where

$$\|f_-; \mathfrak{H}(\mathbb{C}_\alpha^+)\| = \left(\sup_{\psi \in (-\alpha, \pi - \alpha)} \int_0^\infty \|f_-(z_- + e^{i\psi}x); L^2(\Omega)\|^2 dx \right)^{1/2}, \quad e^{i\alpha}z_- \in \mathbb{R}.$$

As a consequence, for all $z \in \mathbb{S}_\alpha$ and $\psi \in (-\alpha, \alpha)$ we have

$$\begin{aligned} \int_0^\infty \|F(z + e^{i\psi}x); L^2(\Omega)\|^2 dx &\leq \|f_+; \mathfrak{H}(\mathbb{C}_\alpha^-)\|^2 + \|f_-; \mathfrak{H}(\mathbb{C}_\alpha^+)\|^2 \\ &= \|F_+; L^2(\mathbb{R}_+ \times \Omega)\|^2 + \|F_-; L^2(\mathbb{R}_+ \times \Omega)\|^2 = \|F; \mathfrak{H}(\mathbb{S}_\alpha)\|^2. \end{aligned}$$

□

Consider the algebra \mathcal{E} of all entire functions $\mathbb{C} \ni z \mapsto F(z) \in C_0^\infty(\Omega)$ with the following property: in any sector $|\Im z| \leq (1 - \epsilon)\Re z$ with $\epsilon > 0$ the value $\|F(z); L^2(\Omega)\|$ decays faster than any inverse power of $\Re z$ as $\Re z \rightarrow +\infty$. Examples of functions $F \in \mathcal{E}$ are $F(z) = e^{-\gamma z^2} P(z)$, where $\gamma > 0$ and $P(z)$ is an arbitrary polynomial in z with coefficients in $C_0^\infty(\Omega)$. Clearly, $\mathcal{E} \subset \mathfrak{H}(\mathbb{S}_\alpha)$. The next lemma is an adaptation of [18, Theorem 3], we omit the proof.

LEMMA 5.2. *The set of functions $\{\mathbb{R}_+ \times \Omega \ni (x, y) \mapsto F \circ \kappa_{\lambda, r}(x, y) : F \in \mathcal{E}\}$ is dense in the space $L^2(\mathbb{R}_+ \times \Omega)$ for any $\lambda \in \mathcal{D}_\alpha$ and $r > 0$. Here $F \circ \kappa_{\lambda, r}(x, \cdot)$ stands for the value of the entire function $z \mapsto F(z)$ at the point $z = x + \lambda s(x - r)$.*

Now we are in position to prove the following proposition.

PROPOSITION 5.3.

1. *The estimate*

$$\int_0^\infty \|F \circ \kappa_{\lambda, r}; L^2(\Omega)\|^2 dx \leq C_r \|F; \mathfrak{H}(\mathbb{S}_\alpha)\|^2, \quad r > 0, \quad (5.5)$$

holds uniformly in $\lambda \in \mathcal{D}_\alpha$ and $F \in \mathfrak{H}(\mathbb{S}_\alpha)$.

2. *The space $\mathfrak{H}(\mathbb{S}_\alpha)$ is sufficiently large in the sense that for any $\lambda \in \mathcal{D}_\alpha$ and $r > 0$ the set $\{\mathbb{R}_+ \times \Omega \ni (x, y) \mapsto F \circ \kappa_{\lambda, r}(x, y) : F \in \mathfrak{H}(\mathbb{S}_\alpha)\}$ is dense in $L^2(\mathbb{R}_+ \times \Omega)$.*
3. *For any $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ and $r > 0$ the function $\mathcal{D}_\alpha \ni \lambda \mapsto F \circ \kappa_{\lambda, r} \in L^2(\mathbb{R}_+ \times \Omega)$ is analytic.*

The proof is preceded by a discussion. By Proposition 5.3.1 for any $r > 0$ the complex scaling $\kappa_{\lambda, r}$ induces the uniformly bounded injection

$$\mathfrak{H}(\mathbb{S}_\alpha) \ni F \mapsto F \circ \kappa_{\lambda, r} \in L^2(\mathbb{R}_+ \times \Omega), \quad \lambda \in \mathcal{D}_\alpha. \quad (5.6)$$

Any function $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ can be reconstructed from its trace $x \mapsto F \circ \kappa_{\lambda,r}(x, \cdot) \in L^2(\Omega)$ by analytic continuation from the curve $\{x + \lambda s(x-r) \in \mathbb{C} : x > 0\}$ to the sector \mathbb{S}_α . Therefore we can always identify the space $\mathfrak{H}(\mathbb{S}_\alpha)$ with the range of injection (5.6). By Proposition 5.3.2 the range is dense in $L^2(\mathbb{R}_+ \times \Omega)$.

Proof. 1. For all $\lambda \in \mathcal{D}_\alpha$ the curve $\{x + \lambda s(x-r) \in \mathbb{C} : x > 0\}$ lies in the sector \mathbb{S}_α . Observe that this curve is differ from a ray only inside an independent of $\lambda \in \mathcal{D}_\alpha$ compact subset $\mathfrak{K}_r \subset \mathbb{S}_\alpha$. Now the uniform in $\lambda \in \mathcal{D}_\alpha$ and $F \in \mathfrak{H}(\mathbb{S}_\alpha)$ estimate (5.5) follows from assertions 3 and 4 of Lemma 5.1.

2. The assertion is an immediate consequence of the embedding $\mathcal{E} \subset \mathfrak{H}(\mathbb{S}_\alpha)$, Lemma 5.2, and the estimate (5.5).

3. It is easy to see that the function $\mathcal{D}_\alpha \ni \lambda \mapsto F \circ \kappa_{\lambda,r} \in L^2(\mathbb{R}_+ \times \Omega)$ is weakly (and therefore strongly) analytic. \square

Introduce the Hilbert space $\mathcal{H}_\alpha(\mathcal{G})$ with the norm

$$\|f; \mathcal{H}_\alpha(\mathcal{G})\| = \|f; L^2(\mathcal{G})\| + \|f \circ \varkappa^{-1}; \mathfrak{H}(\mathbb{S}_\alpha)\|.$$

The space $\mathcal{H}_\alpha(\mathcal{G})$ consists of all functions $f \in L^2(\mathcal{G})$ such that $f \circ \varkappa^{-1}$ is an element of the Hardy space $\mathfrak{H}(\mathbb{S}_\alpha)$; here \varkappa is the diffeomorphism (2.1). As a consequence of Proposition 5.3 and definition (2.5) of the complex scaling $\vartheta_{\lambda,r}$ we immediately get the following assertions.

COROLLARY 5.4.

1. For any $r > 0$ the estimate (5.2) holds with an independent of $f \in \mathcal{H}_\alpha(\mathcal{G})$ and $\lambda \in \mathcal{D}_\alpha$ constant C_r .
2. For any $\lambda \in \mathcal{D}_\alpha$ and $r > 0$ the set $\{f \circ \vartheta_{\lambda,r} \in L^2(\mathcal{G}) : f \in \mathcal{H}_\alpha(\mathcal{G})\}$ is dense in the space $L^2(\mathcal{G})$.
3. For any $f \in \mathcal{H}_\alpha(\mathcal{G})$ and $r > 0$ the function (5.1) is analytic.

6. Limiting absorption principle. Introduce the Sobolev space $H_0^2(\mathcal{G})$ of functions satisfying the homogeneous Dirichlet boundary condition on $\partial\mathcal{G}$ as the completion of the core $C_0^\infty(\overline{\mathcal{G}})$ with respect to the graph norm

$$\|u; H_0^2(\mathcal{G})\| = \|u; L^2(\mathcal{G})\| + \|\Delta u; L^2(\mathcal{G})\|. \quad (6.1)$$

(Integrating by parts and using the Cauchy-Schwarz inequality one can easily see that the norm (6.1) is equivalent to the traditional norm $(\sum_{\ell+|m|\leq 2} \|\partial_\zeta^\ell \partial_\eta^m u; L^2(\mathcal{G})\|^2)^{1/2}$.) By Proposition 4.4.1 the space $H_0^2(\mathcal{G})$ is the domain $D(\Delta)$ of the selfadjoint Dirichlet Laplacian Δ . For the points $\mu \in \sigma(\Delta)$ the resolvent $(\Delta - \mu - ie)^{-1}$ does not have limits in the space of bounded operators $\mathcal{B}(L^2(\mathcal{G}), H_0^2(\mathcal{G}))$ as ϵ tends to zero from below ($\epsilon \uparrow 0$) or from above ($\epsilon \downarrow 0$). However the limits may exist in the space of bounded operators acting from a smaller source space to a larger target space. As a source space we take the Hilbert space $\mathcal{H}_\alpha(\mathcal{G})$ constructed in the previous section. As a target space we take the reflexive Fréchet space $H_{0,\text{loc}}^2(\mathcal{G})$. The space $H_{0,\text{loc}}^2(\mathcal{G})$ consists of all distributions u such that $\varrho u \in H_0^2(\mathcal{G})$ with any $\varrho \in C_c^\infty(\overline{\mathcal{G}})$, the topology in $H_{0,\text{loc}}^2(\mathcal{G})$ is induced by the family of seminorms $u \mapsto \|\varrho u; H_0^2(\mathcal{G})\|$; here $C_c^\infty(\overline{\mathcal{G}})$ is the set of all smooth functions with compact supports in $\overline{\mathcal{G}}$.

THEOREM 6.1. *Let $\sigma(\Delta_\Omega)$ stand for the spectrum of the selfadjoint Dirichlet Laplacian Δ_Ω in $L^2(\Omega)$. Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is not an eigenvalue of the selfadjoint Dirichlet Laplacian Δ in $L^2(\mathcal{G})$. Then the following assertions hold.*

1. For all sufficiently large $r > 0$ and $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$ the resolvent

$$(\Delta_{\lambda,r} - \mu_0)^{-1} : L^2(\mathcal{G}) \rightarrow H_0^2(\mathcal{G})$$

is a bounded operator.

2. The resolvent $(\Delta - \mu_0 - i\epsilon)^{-1}$, $\epsilon \geq 0$, viewed as a bounded operator acting from $\mathcal{H}_\alpha(\mathcal{G})$ to $H_{0,\text{loc}}^2(\mathcal{G})$, has limits as $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$.
3. Suppose that $r > 0$ is sufficiently large, $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$, and $f \in \mathcal{H}_\alpha(\mathcal{G})$. Let $u_{\lambda,r} \in H_0^2(\mathcal{G})$ be given by the equality $u_{\lambda,r} = (\Delta_{\lambda,r} - \mu_0)^{-1}(f \circ \vartheta_{\lambda,r})$. Then the outgoing $u_- \in H_{0,\text{loc}}^2(\mathcal{G})$ and the incoming $u_+ \in H_{0,\text{loc}}^2(\mathcal{G})$ solutions defined by the limiting absorption principle

$$u_+ = \lim_{\epsilon \uparrow 0} (\Delta - \mu_0 - i\epsilon)^{-1} f, \quad u_- = \lim_{\epsilon \downarrow 0} (\Delta - \mu_0 - i\epsilon)^{-1} f, \quad (6.2)$$

meet the relation

$$u_{\lambda,r} \upharpoonright_{\mathcal{G}_r} = \begin{cases} u_+ \upharpoonright_{\mathcal{G}_r}, & \Im \lambda < 0; \\ u_- \upharpoonright_{\mathcal{G}_r}, & \Im \lambda > 0. \end{cases}$$

Here the bounded domain \mathcal{G}_r is the same as in (2.9).

The proof is preceded by a discussion. From Theorem 6.1.3 we see that the equation $(\Delta_{\lambda,r} - \mu_0)u_{\lambda,r} = f \circ \vartheta_{\lambda,r}$ with non-real parameter $\lambda \in \mathcal{D}_\alpha$ describes infinite PMLs on $\mathcal{G} \setminus \mathcal{G}_r$ for all μ_0 satisfying the assumptions of the theorem. The layers are perfectly matched in the sense that for $\Im \lambda > 0$ (resp. for $\Im \lambda < 0$) $u_{\lambda,r}$ coincides in \mathcal{G}_r with the outgoing solution u_- (resp. with the incoming solution u_+). The PMLs are absorbing because in contrast to u_\pm the function $u_{\lambda,r}$ decays at infinity in the mean as an element of $H_0^2(\mathcal{G})$. In the next section we will refine results of Theorem 6.1 by showing that under an additional assumption on $f \circ \vartheta_{\lambda,r}$ the solution $u_{\lambda,r}$ is of some exponential decay at infinity. For instance, this assumption is a priori met for $f \in L^2(\mathcal{G})$ supported in \mathcal{G}_r . Then $f \circ \vartheta_{\lambda,r} \equiv f$ and the operator $\Delta_{\lambda,r}$ completely describes infinite PMLs on $\mathcal{G} \setminus \mathcal{G}_r$.

Proof. The proof consists of two steps.

Step 1. In Section 8 below we will show that the graph norm of $\Delta_{\lambda,r}$ is an equivalent norm in $H_0^2(\mathcal{G})$. This immediately implies that $D(\Delta_{\lambda,r}) = H_0^2(\mathcal{G})$ as the set $C_0^\infty(\overline{\mathcal{G}})$ is dense in both spaces, cf. Proposition 4.4.1. We will also localize the essential spectrum $\sigma_{ess}(\Delta_{\lambda,r})$ of the unbounded m -sectorial operator $\Delta_{\lambda,r}$ in $L^2(\mathcal{G})$. As is well-known, the spectrum $\sigma(\Delta_\Omega)$ consists of infinitely many positive isolated eigenvalues. It turns out that $\sigma_{ess}(\Delta_{\lambda,r})$ consists of an infinite number of rays emanating from every point $\nu \in \sigma(\Delta_\Omega)$, cf. Figure 6.1. By definition $\sigma(\Delta_\Omega)$ is the set of thresholds of

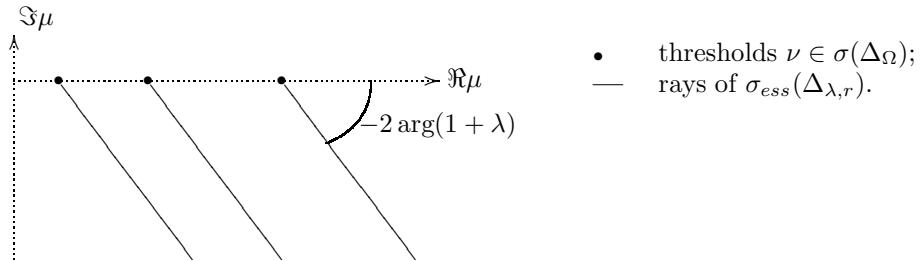


FIG. 6.1. Essential spectrum of the m -sectorial operator $\Delta_{\lambda,r}$ for $\Im \lambda > 0$.

the Dirichlet Laplacian Δ . As λ varies, the ray $\{\mu \in \mathbb{C} : \arg(\mu - \nu) = -2 \arg(1 + \lambda)\}$ of the essential spectrum $\sigma_{ess}(\Delta_{\lambda,r})$ rotates about the threshold $\nu \in \sigma(\Delta_\Omega)$ and sweeps

the sector $\{\mu \in \mathbb{C} : |\arg(\mu - \nu)| < 2\alpha\}$. In order to avoid several repetitions we organized the paper so that the proofs of these results in some greater generality are postponed to Section 8; here we take these results for granted.

Recall that μ is said to be a point of the essential spectrum $\sigma_{ess}(A)$ of a closed unbounded operator A in the space $L^2(\mathcal{G})$ with the domain $H_0^2(\mathcal{G})$, if the bounded operator $A - \mu : H_0^2(\mathcal{G}) \rightarrow L^2(\mathcal{G})$ is not Fredholm (a linear bounded operator between two Banach spaces is Fredholm, if its kernel and cokernel are finite dimensional, and its range is closed). Since the operator $\Delta_{\lambda,r}$ is m-sectorial, there exists a regular point of $\Delta_{\lambda,r}$ in the simply connected set $\mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r})$. Therefore

$$\mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r}) \ni \mu \mapsto \Delta_{\lambda,r} - \mu : H_0^2(\mathcal{G}) \rightarrow L^2(\mathcal{G})$$

is a Fredholm holomorphic operator function, e.g. [24, Appendix]. Recall that the spectrum of a Fredholm holomorphic operator function consists of isolated eigenvalues of finite algebraic multiplicity, e.g. [24, Proposition A.8.4]. As a consequence, the resolvent

$$\mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r}) \ni \mu \mapsto (\Delta_{\lambda,r} - \mu)^{-1} : L^2(\mathcal{G}) \rightarrow H_0^2(\mathcal{G}) \quad (6.3)$$

is a meromorphic operator function.

For $f, g \in L^2(\mathcal{G})$ and a sufficiently large $r > 0$ we define the quadratic form

$$(f, g)_{\lambda,r} = \int_{\mathcal{G}} f \bar{g} \sqrt{\det \mathbf{e}_{\lambda,r}} d\zeta d\eta, \quad \lambda \in \mathcal{D}_\alpha.$$

This form is bounded in $L^2(\mathcal{G})$. Indeed, thanks to the estimate (3.7) on $\mathbf{e}_{\lambda,r}^{-1}$ we have $0 < c_1 \leq |\det \mathbf{e}_{\lambda,r}(\zeta, \eta)| \leq c_2$ uniformly in $\lambda \in \mathcal{D}_\alpha$ and $(\zeta, \eta) \in \overline{\mathcal{G}}$.

Assume that $\lambda \in \mathcal{D}_\alpha$ is real. Then the form $(\cdot, \cdot)_{\lambda,r}$ is the inner product induced on \mathcal{G} by the metric $\mathbf{e}_{\lambda,r}$, the norm $\sqrt{(f, f)_{\lambda,r}}$ is equivalent to the norm $\|f; L^2(\mathcal{G})\|$, and $\Delta_{\lambda,r}$ is the Laplace-Beltrami operator on $(\mathcal{G}, \mathbf{e}_{\lambda,r})$. We have

$$(\Delta - \mu)u = ((\Delta_{\lambda,r} - \mu)(u \circ \vartheta_{\lambda,r})) \circ \vartheta_{\lambda,r}^{-1} \quad \forall u \in C_0^\infty(\overline{\mathcal{G}}). \quad (6.4)$$

Assume that μ is not in the sector of $\Delta_{\lambda,r}$. Then $(\Delta_{\lambda,r} - \mu)^{-1}$ is a bounded operator and we can rewrite (6.4) in the form

$$(\Delta - \mu)^{-1}f = ((\Delta_{\lambda,r} - \mu)^{-1}(f \circ \vartheta_{\lambda,r})) \circ \vartheta_{\lambda,r}^{-1}, \quad (6.5)$$

where f is in the set $\{f = (\Delta - \mu)u : u \in C_0^\infty(\overline{\mathcal{G}})\}$. This set is dense in $L^2(\mathcal{G})$, because $C_0^\infty(\overline{\mathcal{G}})$ is dense in $H_0^2(\mathcal{G})$, and the operator $\Delta - \mu : H_0^2(\mathcal{G}) \rightarrow L^2(\mathcal{G})$ yields an isomorphism. It is clear that $(f \circ \vartheta_{\lambda,r}, f \circ \vartheta_{\lambda,r})_{\lambda,r} = (f, f)$. As a consequence, the (real) scaling $f \mapsto f \circ \vartheta_{\lambda,r}$ realizes an isomorphism in $L^2(\mathcal{G})$, and the equality (6.5) extends by continuity to all $f \in L^2(\mathcal{G})$. Taking the inner product of the equality (6.5) with $g \in L^2(\mathcal{G})$, and passing to the variables $(\tilde{\zeta}, \tilde{\eta}) = \vartheta_\lambda(\zeta, \eta)$ in the right hand side, we obtain

$$((\Delta - \mu)^{-1}f, g) = ((\Delta_{\lambda,r} - \mu)^{-1}(f \circ \vartheta_{\lambda,r}), g \circ \vartheta_{\lambda,r})_{\lambda,r}. \quad (6.6)$$

Now we assume that $f, g \in \mathcal{H}_\alpha(\mathcal{G})$. Then $f \circ \vartheta_{\lambda,r}$ and $g \circ \vartheta_{\lambda,r}$ are $L^2(\mathcal{G})$ -valued analytic functions of λ in the disk \mathcal{D}_α , see Corollary 5.4.3. This together with Proposition 4.4.4 implies that the right hand side of (6.6) extends by analyticity from $\lambda \in \mathcal{D}_\alpha \cap \mathbb{R}$ to all

$\lambda \in \mathcal{D}_\alpha$. The right hand side of (6.6) extends from all μ outside of the sector of $\Delta_{\lambda,r}$ to a meromorphic function of $\mu \in \mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r})$. In particular, for all $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$ we have $\mu_0 \in \mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r})$, cf. Figure 6.1. Here μ_0 is the same as in the formulation of the theorem.

Now we are in position to prove assertion 1. Consider the projection

$$P = \text{s-lim}_{\epsilon \downarrow 0} i\epsilon(\Delta - \mu_0 - i\epsilon)^{-1}$$

onto the eigenspace of the selfadjoint operator Δ . Suppose, by contradiction, that the resolvent (6.3) has a pole at the point $\mu_0 \in \mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r})$. By Corollary 5.4.2 there exist f and g in the space $\mathcal{H}_\alpha(\mathcal{G})$ such that μ_0 is a pole of the right hand side of (6.6). The equality (6.6) implies that $(Pf, g) \neq 0$, and thus $\ker(\Delta - \mu_0) \neq \{0\}$. This is a contradiction. Assertion 1 is proven.

Step 2. We need to show that for any $\varrho \in C_c^\infty(\overline{\mathcal{G}})$ the operator $\varrho(\Delta - \mu_0 - i\epsilon)^{-1}$ tends to some limits in the space of bounded operators $\mathcal{B}(\mathcal{H}_\alpha(\mathcal{G}), H_0^2(\mathcal{G}))$ as $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$. We take a sufficiently large $r = r(\varrho) > 0$ such that $\text{supp } \varrho \subset \overline{\mathcal{G}_r}$. Then $\varrho \circ \vartheta_{\lambda,r} = \varrho$ for all $\lambda \in \mathcal{D}_\alpha$. Now we can pass from (6.5) to the equality

$$\varrho(\Delta - \mu)^{-1}f = \varrho(\Delta_{\lambda,r} - \mu)^{-1}(f \circ \vartheta_{\lambda,r}). \quad (6.7)$$

For $f \in \mathcal{H}_\alpha(\mathcal{G})$ the equality (6.7) extends by analyticity to all $\lambda \in \mathcal{D}_\alpha$. Consider, for instance, the case $\Im \lambda > 0$ (the case $\Im \lambda < 0$ is similar). Then the upper half-plane $\mathbb{C}^+ = \{\mu \in \mathbb{C} : \Im \mu > 0\}$ and a complex neighborhood of the point μ_0 do not contain points of $\sigma_{ess}(\Delta_{\lambda,r})$, cf. Figure 6.1. Therefore the right hand side of (6.7) has a meromorphic continuation in μ to the union of \mathbb{C}^+ and a complex neighborhood of μ_0 . Hence the left hand side of (6.7) has the same meromorphic continuation. Clearly, a pole at μ_0 may only appear due to a pole of the resolvent (6.3) at μ_0 , but it is a regular point by assertion 1. Since ϱ is an arbitrary smooth function supported in \mathcal{G}_r this proves assertion 3. In order to prove assertion 2 it remains to note that

$$\begin{aligned} \|\lim_{\epsilon \downarrow 0} \varrho(\Delta - \mu_0 - i\epsilon)^{-1}f; H_0^2(\mathcal{G})\| &= \|\varrho(\Delta_{\lambda,r} - \mu_0)^{-1}(f \circ \vartheta_{\lambda,r}); H_0^2(\mathcal{G})\| \\ &\leq C(\varrho) \|(\Delta_{\lambda,r} - \mu_0)^{-1}; L^2(\mathcal{G}) \rightarrow H_0^2(\mathcal{G})\| \|f \circ \vartheta_{\lambda,r}; L^2(\mathcal{G})\| \\ &\leq C(\varrho, r, \mu_0) \|f; \mathcal{H}_\alpha(\mathcal{G})\|. \end{aligned}$$

In the last inequality we used Corollary 5.4.1 and assertion 1. \square

In the following two remarks we collect some results that can be obtained by methods developed in the proof of Theorem 6.1. Although these results are not used in this paper, they provide additional insights of the problem.

REMARK 6.2. *On the basis of the equality (6.6), Corollary 5.4, and the description of $\sigma_{ess}(\Delta_{\lambda,r})$ for $\lambda \in \mathcal{D}_\alpha$, one can develop an analog of the celebrated Aguilar-Balslev-Combes-Simon theory of resonances [10, 16, 18, 34]. We announce some results below, for the proof we refer to [21].*

1. *The selfadjoint Dirichlet Laplacian Δ in \mathcal{G} has no singular continuous spectrum, its eigenvalues can accumulate only at thresholds $\nu \in \sigma(\Delta_\Omega)$. (Examples of accumulating eigenvalues can be found e.g. in [13].)*
2. *The spectrum $\sigma(\Delta_{\lambda,r})$ of the m -sectorial operator $\Delta_{\lambda,r}$ does not depend on the choice of the scaling function \underline{s} satisfying (2.2)–(2.4). Moreover, the spectrum $\sigma(\Delta_{\lambda,r})$ lies in the half-plane \mathbb{C}^+ in the case $\Im \lambda \geq 0$ and $\sigma(\Delta_{\lambda,r}) \subset \overline{\mathbb{C}^-}$ in the case $\Im \lambda \leq 0$, where $\mathbb{C}^\pm = \{\mu \in \mathbb{C} : \Im \mu \gtrless 0\}$ and $\lambda \in \mathcal{D}_\alpha$.*

3. A point $\mu \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is an eigenvalue of Δ if and only if it is an isolated eigenvalue of $\Delta_{\lambda,r}$, where $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$ and $r > 0$ is sufficiently large.
4. The resolvent matrix elements $((\Delta - \mu)^{-1} f, g)$, where $f, g \in \mathcal{H}_\alpha(\mathcal{G})$, have meromorphic continuations from the physical sheet $\mathbb{C} \setminus \sigma_{ess}(\Delta)$ across $\sigma_{ess}(\Delta)$ to the set $\mathbb{C} \setminus \sigma_{ess}(\Delta_{\lambda,r})$. Moreover, μ is a pole of the continuation for some $f, g \in \mathcal{H}_\alpha(\mathcal{G})$ if and only if it is an isolated eigenvalue of $\Delta_{\lambda,r}$. The non-real isolated eigenvalues of $\Delta_{\lambda,r}$ are naturally identified with resonances of Δ .
5. As λ changes continuously in the disk \mathcal{D}_α , an isolated eigenvalue of $\Delta_{\lambda,r}$ survives while it is not covered by one of the rotating rays of $\sigma_{ess}(\Delta_{\lambda,r})$.

REMARK 6.3. The argument of the second step in the proof of Theorem 6.1 allows also to see that the resolvent $(\Delta - \mu)^{-1} : \mathcal{H}_\alpha(\mathcal{G}) \rightarrow H_{0,loc}^2(\mathcal{G})$ has a meromorphic continuation from the physical sheet $\mathbb{C} \setminus \sigma_{ess}(\Delta)$ across the intervals (ν_-, ν_+) between the neighboring thresholds $\nu_\pm \in \sigma(\Delta_\Omega)$ to a Riemann surface. The surface consists of the physical sheet $\mathbb{C} \setminus \sigma_{ess}(\Delta)$ of the Dirichlet Laplacian and an infinite number of the sectors $\{\mu \in \mathbb{C} : 0 > \arg(\mu - \nu_-) > -2\alpha\}$ attached to $\mathbb{C}^+ \subset \mathbb{C} \setminus \sigma_{ess}(\Delta)$ and of the sectors $\{\mu \in \mathbb{C} : 0 < \arg(\mu - \nu_-) < 2\alpha\}$ attached to $\mathbb{C}^- \subset \mathbb{C} \setminus \sigma_{ess}(\Delta)$ along the intervals (ν_-, ν_+) . Indeed, for any $\varrho \in C_c^\infty(\overline{\mathcal{G}})$ we can take a sufficiently large $r = r(\varrho) > 0$ such that $\text{supp } \varrho \subset \overline{\mathcal{G}_r}$. As λ varies in $\mathcal{D}_\alpha \cap \mathbb{C}^+$ (resp. in $\mathcal{D}_\alpha \cap \mathbb{C}^-$) the strip between the neighboring rays $\{\mu \in \mathbb{C} : \arg(z - \nu_\pm) = -2\arg(1 + \lambda)\}$ of $\sigma_{ess}(\Delta_{\lambda,r})$ sweeps the sector $\{\mu \in \mathbb{C} : 0 > \arg(\mu - \nu_-) > -2\alpha\}$ (resp. the sector $\{\mu \in \mathbb{C} : 0 < \arg(\mu - \nu_-) < 2\alpha\}$) and the right hand side of (6.7) provides the left hand side with a meromorphic continuation to the strip. The poles of the continuation are resonances of the Dirichlet Laplacian Δ [39].

The results listed in Remarks 6.2 and 6.3 are new and might be of their own interest. Traditionally, when studying Laplacians in the waveguide-type of geometry, one imposes more or less restrictive assumptions on the rate of convergence of the metric g on $(0, \infty) \times \overline{\Omega}$ to its limit at infinity; see, e.g., [9, 11, 12, 13, 14, 15, 19, 29, 30, 31]. Contrastingly, our assumptions on the diffeomorphism \varkappa allow for arbitrarily slow convergence of the metric $g = \varkappa^* e$ to the Euclidean metric e at infinity, see Section 3. As a substitution for the assumptions on the rate of convergence of the metric g at infinity we use assumptions on the analytic regularity of the diffeomorphism \varkappa .

Let us also mention here the paper [33], where general elliptic problems whose coefficients slowly converge to their limits at infinity are considered. It is shown that any finite accumulation point of eigenvalues corresponding to exponentially decaying eigenfunctions is a threshold, these accumulations are characterized in terms of some non-classical ‘‘augmented scattering matrices.’’ An additional investigation on decay of eigenfunctions at infinity is required in order to say whether these results describe accumulations of eigenvalues of the Dirichlet Laplacian Δ or not. This goes beyond the scope of the present paper, we refer to [22].

7. Exponential decay of solutions in infinite PMLs. In this section we prove the following theorem.

THEOREM 7.1. *Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is not an eigenvalue of the selfadjoint Dirichlet Laplacian Δ in $L^2(\mathcal{G})$, the scaling parameter $\lambda \in \mathcal{D}_\alpha$ is not real, and*

$$0 \leq \beta < \min_{\nu \in \sigma(\Delta_\Omega)} |\Im\{(1 + \lambda)\sqrt{\mu_0 - \nu}\}|. \quad (7.1)$$

Let $s(\zeta, \eta)$ stand for the scaling function $\mathbb{R}_+ \times \overline{\Omega} \ni (x, y) \mapsto s(x)$ written in the coordinates $(\zeta, \eta) \in \overline{\mathcal{C}}$ and extended to $\overline{\mathcal{G}}$ by zero; see (2.2)–(2.4). Then for all sufficiently large $r > 0$ and all $\mathcal{F} \in L^2(\mathcal{G})$ satisfying $e^{\beta s} \mathcal{F} \in L^2(\mathcal{G})$ the estimate

$$\|e^{\beta s}(\Delta_{\lambda,r} - \mu_0)^{-1} \mathcal{F}; H_0^2(\mathcal{G})\| \leq C \|e^{\beta s} \mathcal{F}; L^2(\mathcal{G})\| \quad (7.2)$$

is valid with a constant C independent of \mathcal{F} .

Theorem 7.1 together with Theorem 6.1 shows that under the additional assumption $e^{\beta s}(f \circ \vartheta_{\lambda,r}) \in L^2(\mathcal{G})$ on $f \in \mathcal{H}_\alpha(\mathcal{G})$ infinite PMLs absorb outgoing or incoming solutions (depending on the sign of $\Im \lambda$) so effectively that the function $u_{\lambda,r}$ in Theorem 7.1.3 exponentially decays at infinity in the mean.

Proof. Consider the conjugated operator $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$ as an unbounded operator in $L^2(\mathcal{G})$ with the domain $C_0^\infty(\overline{\mathcal{G}})$. With this operator we associate the quadratic form $q_{\lambda,r}^\beta[u, u] = (e^{\beta s} \Delta_{\lambda,r} e^{-\beta s} u, u)_{\lambda,r}$. Observe that

$$\begin{aligned} q_{\lambda,r}^\beta[u, u] - q_{\lambda,r}[u, u] &= -\beta^2 \int_{\mathcal{G}} \langle \mathbf{e}_{\lambda,r}^{-1} u \nabla_{\zeta\eta} \mathbf{s}, u \nabla_{\zeta\eta} \mathbf{s} \rangle d\zeta d\eta \\ &\quad - \beta \int_{\mathcal{G}} \langle (\det \mathbf{e}_{\lambda,r})^{1/2} \mathbf{e}_{\lambda,r}^{-1} u \nabla_{\zeta\eta} \mathbf{s}, \nabla_{\zeta\eta} (\det \mathbf{e}_{\lambda,r})^{-1/2} u \rangle d\zeta d\eta \\ &\quad + \beta \int_{\mathcal{G}} \langle \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u \nabla_{\zeta\eta} \mathbf{s} \rangle d\zeta d\eta, \end{aligned}$$

where $q_{\lambda,r}$ is the same as in (4.1). Since the right hand side depends linearly on $\nabla_{\zeta\eta} u$, similarly to the proof of Lemma 4.3 one can deduce

$$|q_{\lambda,r}^\beta[u, u] - q_{\lambda,r}[u, u]| \leq \epsilon |(-\nabla_{\zeta\eta} \cdot \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} u, u)| + C\epsilon^{-1} \|u; L^2(\mathcal{G})\|^2$$

with an arbitrary small $\epsilon > 0$ and a constant C independent of $u \in C_0^\infty(\overline{\mathcal{G}})$. This estimate together with (4.8), (4.9), and Lemma 4.1 implies that for all sufficiently large $r > 0$ we have

$$|\arg(q_{\lambda,r}^\beta[u, u] + \gamma \|u; L^2(\mathcal{G})\|^2)| \leq \varphi \quad (7.3)$$

with some angle $\varphi < \pi/2$ and $\gamma > 0$, which are independent of $u \in C_0^\infty(\overline{\mathcal{G}})$. Therefore $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$ with the domain $C_0^\infty(\overline{\mathcal{G}})$ is a densely defined sectorial operator in $L^2(\mathcal{G})$. Let $D(e^{\beta s} \Delta_{\lambda,r} e^{-\beta s})$ be the domain of its m-sectorial Friedrichs extension [23, Chapter VI.2]. The Friedrichs extension will also be denoted by $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$.

As in the proof of Proposition 4.4.1 we conclude that $C_0^\infty(\overline{\mathcal{G}})$ is a core of the m-sectorial operator $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$. In Section 8 we will show that the graph norm of $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$ is equivalent to the norm in $H_0^2(\mathcal{G})$; hence $D(e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}) = H_0^2(\mathcal{G})$. Furthermore, we will localize the essential spectrum $\sigma_{ess}(e^{\beta s} \Delta_{\lambda,r} e^{-\beta s})$ of the m-sectorial operator $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$, see Proposition 8.1. It turns out that the essential spectrum consists of an infinite number of parabolas, see Figure 7.1. In the case $\beta = 0$ the parabolas collapse to the dashed rays originating from the thresholds $\nu \in \sigma(\Delta_\Omega)$ and we obtain the essential spectrum $\sigma_{ess}(\Delta_{\lambda,r})$.

The m-sectorial operator $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$ defines the Fredholm holomorphic operator function

$$\mu \mapsto e^{\beta s} \Delta_{\lambda,r} e^{-\beta s} - \mu : H_0^2(\mathcal{G}) \rightarrow L^2(\mathcal{G})$$

on the simply connected subset of $\mathbb{C} \setminus \sigma_{ess}(e^{\beta s} \Delta_{\lambda,r} e^{-\beta s})$ containing an infinite part of the real negative semiaxis (regular points of $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$). Condition (7.1) on β guarantees that the point μ_0 is in this simply connected subset. As the spectrum of a Fredholm holomorphic operator function consists of isolated eigenvalues of finite multiplicity, μ_0 is a regular point or an eigenvalue of $e^{\beta s} \Delta_{\lambda,r} e^{-\beta s}$. The inclusion $\Psi \in \ker(e^{\beta s} \Delta_{\lambda,r} e^{-\beta s} - \mu_0)$ implies $e^{-\beta s} \Psi \in \ker(\Delta_{\lambda,r} - \mu_0)$, and hence $\Psi \equiv 0$ by

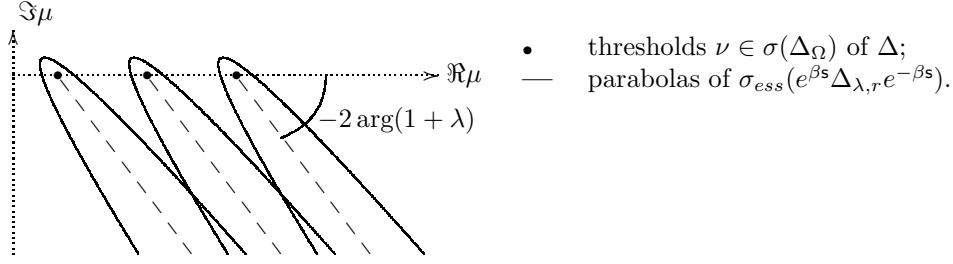


FIG. 7.1. Essential spectrum of the conjugated operator $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}$ for $\Im \lambda > 0$ and $\beta \geq 0$.

Theorem 6.1.1. Thus the operator $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu_0$ yields an isomorphism between the spaces $H_0^2(\mathcal{G})$ and $L^2(\mathcal{G})$. This together with the equality

$$(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu_0)^{-1} e^{\beta s} \mathcal{F} = e^{\beta s} (\Delta_{\lambda, r} - \mu_0)^{-1} \mathcal{F}, \quad e^{\beta s} \mathcal{F} \in L^2(\mathcal{G}),$$

justifies the estimate (7.2). \square

8. Localization of the essential spectrum. In this section we localize the essential spectrum $\sigma_{ess}(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$ of the m-sectorial operator $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}$ in $L^2(\mathcal{G})$ with parameters $\lambda \in \mathcal{D}_\alpha$ and $\beta \in \mathbb{R}$; here s is the same as in Theorem 7.1. In particular, in the case $\beta = 0$ we find $\sigma_{ess}(\Delta_{\lambda, r})$. We also prove that $D(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}) = H_0^2(\mathcal{G})$. In other words, we show that the information on the essential spectrum and the domain of $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}$ we used in the proofs of Theorem 6.1 and Theorem 7.1 is correct.

Let us note that for fixed λ and β the spectrum $\sigma_{ess}(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$ depends only on the behavior of the scaling function s and the matrix $e_{\lambda, r}$ outside any compact region of \mathcal{G} . In order to control $\sigma_{ess}(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$ we imposed the condition (2.4).

PROPOSITION 8.1. *Assume that $\lambda \in \mathcal{D}_\alpha$, $\beta \in \mathbb{R}$, and $r > 0$ is sufficiently large. Then the following assertions hold.*

1. *The Hilbert spaces $D(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$ and $H_0^2(\mathcal{G})$ are coincident and their norms are equivalent.*
2. *The bounded operator*

$$e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu : H_0^2(\mathcal{G}) \rightarrow L^2(\mathcal{G}) \quad (8.1)$$

is not Fredholm (or, equivalently, $\mu \in \sigma_{ess}(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$) if and only if the parameters μ , λ , and β meet the condition

$$\nu - \mu = (1 + \lambda)^{-2}(\beta + i\xi)^2 \text{ for some } \nu \in \sigma(\Delta_\Omega) \text{ and } \xi \in \mathbb{R}. \quad (8.2)$$

Proof. The proof is essentially based on methods of the theory of elliptic non-homogeneous boundary value problems [24, 25, 28, 26]. We will rely on the following lemma due to Peetre, see e.g. [26, Lemma 5.1], [25, Lemma 3.4.1] or [32]:

Let \mathcal{X}, \mathcal{Y} and \mathcal{Z} be Banach spaces, where \mathcal{X} is compactly embedded into \mathcal{Z} . Furthermore, let \mathcal{L} be a linear bounded operator from \mathcal{X} to \mathcal{Y} . Then the next two assertions are equivalent: (i) the range of \mathcal{L} is closed in \mathcal{Y} and $\dim \ker \mathcal{L} < \infty$, (ii) there exists a constant C , such that

$$\|u; \mathcal{X}\| \leq C(\|\mathcal{L}u; \mathcal{Y}\| + \|u; \mathcal{Z}\|) \quad \forall u \in \mathcal{X}. \quad (8.3)$$

Below we assume that μ , λ , and β does not meet the condition (8.2) and establish the coercive estimate

$$\|u; H_0^2(\mathcal{G})\| \leq C(\|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu)u; L^2(\mathcal{G})\| + \|w u; L^2(\mathcal{G})\|) \quad \forall u \in H_0^2(\mathcal{G}). \quad (8.4)$$

Here $w \in C^\infty(\overline{\mathcal{G}})$ is a positive rapidly decreasing at infinity weight, such that the embedding of $H_0^2(\mathcal{G})$ into the weighted space $L^2(\mathcal{G}; w)$ with the norm $\|w \cdot; L^2(\mathcal{G})\|$ is compact. Note that (8.4) is an estimate of type (8.3) for the operator (8.1).

The strongly elliptic differential operator $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}$ endowed with the Dirichlet boundary condition set up a regular elliptic boundary value problem. Solutions of a regular elliptic boundary value problem satisfy local coercive estimates, e.g. [26] or [25]. Thus we have the local coercive estimate

$$\|\rho_T u; H_0^2(\mathcal{G})\| \leq C(\|\varrho_T(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu)u; L^2(\mathcal{G})\| + \|\varrho_T u; L^2(\mathcal{G})\|). \quad (8.5)$$

Here ρ_T and ϱ_T are smooth compactly supported cutoff functions in \mathcal{G} such that $\rho_T(\zeta, \eta) = 1$ for $|\zeta| < T + 1$ and $\varrho_T \rho_T = \rho_T$, where T is a large fixed number.

Let $\chi_T \in C^\infty(\overline{\mathcal{G}})$ be another cutoff function such that $\chi_T(\zeta, \eta) = 1$ for $|\zeta| > T$ and $\chi_T(\zeta, \eta) = 0$ for $|\zeta| < T - 1$. On the next step we establish the estimate (8.4) with u replaced by $\chi_T u$. We will do it in the coordinates $(x, y) \in \mathbb{R}_+ \times \overline{\Omega}$.

Let $L^2(\mathbb{R} \times \Omega)$ be the space of functions in the infinite cylinder $\mathbb{R} \times \Omega$ with the norm $(\int_{\mathbb{R}} \|u(x); L^2(\Omega)\|^2 dx)^{1/2}$. Introduce the Sobolev space $H_0^2(\mathbb{R} \times \Omega)$ of functions with zero Dirichlet data on $\mathbb{R} \times \partial\Omega$ as the completion of the set $C_0^\infty(\mathbb{R} \times \overline{\Omega})$ with respect to the norm

$$\|u; H_0^2(\mathbb{R} \times \Omega)\| = \left(\sum_{\ell+|m|\leq 2} \|\partial_x^\ell \partial_y^m u; L^2(\mathbb{R} \times \Omega)\|^2 \right)^{1/2}.$$

Denote $u = (\chi_T u) \circ \varkappa$, where \varkappa is the diffeomorphism (2.1). Let

$$\Delta_{\lambda, r} = -(\det g_{\lambda, r})^{-1/2} \nabla_{xy} \cdot (\det g_{\lambda, r})^{1/2} g_{\lambda, r}^{-1} \nabla_{xy}, \quad \lambda \in \mathcal{D}_\alpha, \quad (8.6)$$

be the operator $\Delta_{\lambda, r}$ written in the coordinates (x, y) . Here $g_{\lambda, r}$ is the matrix (3.3) and $\nabla_{xy} = (\partial_x, \partial_{y_1} \dots \partial_{y_n})^\top$. Due to our assumptions on \varkappa the estimates $0 < \epsilon \leq \det \varkappa(x, y) \leq 1/\epsilon$ hold uniformly in $(x, y) \in \mathbb{R}_+ \times \overline{\Omega}$. Hence for some independent of $u \in C_0^\infty(\overline{\mathcal{G}})$ constants c_1, c_2 , and c_3 we have

$$\begin{aligned} \|\chi_T u; H_0^2(\mathcal{G})\| &= \|\Delta(\chi_T u); L^2(\mathcal{G})\| + \|\chi_T u; L^2(\mathcal{G})\| \\ &\leq c_1(\|\Delta_{0, r} u; L^2(\mathbb{R} \times \Omega)\| + \|u; L^2(\mathbb{R} \times \Omega)\|) \leq c_2 \|u; H_0^2(\mathbb{R} \times \Omega)\|, \\ \|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu)u; L^2(\mathbb{R} \times \Omega)\| &\leq c_3 \|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) \chi_T u; L^2(\mathcal{G})\|. \end{aligned} \quad (8.7)$$

Here the functions u , s , and $\Delta_{\lambda, r} u \equiv (\Delta_{\lambda, r}(\chi_T u)) \circ \varkappa$ are extended from $\mathbb{R}_+ \times \overline{\Omega}$ to the infinite cylinder $\mathbb{R} \times \overline{\Omega}$ by zero, and $\|\Delta_{0, r} u; L^2(\mathbb{R} \times \Omega)\| \leq C \|u; H_0^2(\mathbb{R} \times \Omega)\|$ because the coefficients of the Laplacian $\Delta_{0, r}$ are bounded, cf. (8.6) and (3.5). As T is large, the function u is supported in a small neighborhood of infinity. Due to the stabilization condition (3.4) on $g_{\lambda, r}^{-1}$ and the condition (2.4) on the scaling function s the coefficients of the differential operator

$$e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \Delta_\Omega + (1 + \lambda)^{-2} (\partial_x + \beta)^2$$

are small on the support of u . As a result we get the estimate

$$\|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \Delta_\Omega + (1 + \lambda)^{-2}(\partial_x + \beta)^2)u; L^2(\mathbb{R} \times \Omega)\| \leq \epsilon \|u; H_0^2(\mathbb{R} \times \Omega)\|, \quad (8.8)$$

where ϵ is small and independent of $u \in C_0^\infty(\bar{\mathcal{G}})$; moreover, $\epsilon \rightarrow 0$ as $T \rightarrow +\infty$.

Consider the bounded operator

$$\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu : H_0^2(\mathbb{R} \times \Omega) \rightarrow L^2(\mathbb{R} \times \Omega). \quad (8.9)$$

Applying the Fourier transform $\mathcal{F}_{x \mapsto \xi}$ we pass from the operator (8.9) to the selfadjoint Dirichlet Laplacian $\Delta_\Omega + (1 + \lambda)^{-2}(\beta + i\xi)^2 - \mu$ in $L^2(\Omega)$. Since μ , λ , and β do not meet the condition (8.2), the spectral parameter $\mu - (1 + \lambda)^{-2}(\beta + i\xi)^2$ is outside of the spectrum of Δ_Ω for all $\xi \in \mathbb{R}$. Then a known argument [25, Theorem 5.2.2], [24, Theorem 2.4.1], which is also used as a part of the proof of Lemma 9.2 below, implies that the operator (8.9) realizes an isomorphism. In particular, the estimate

$$\|u; H_0^2(\mathbb{R} \times \Omega)\| \leq c \|\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu)u; L^2(\mathbb{R} \times \Omega)\|$$

is valid with an independent of $u \in H_0^2(\mathbb{R} \times \Omega)$ constant c . As a consequence of this estimate and (8.8) we obtain

$$\begin{aligned} (1 - \epsilon c) \|u; H_0^2(\mathbb{R} \times \Omega)\| &\leq c \|\Delta_\Omega - (1 + \lambda)^{-2}(\partial_x + \beta)^2 - \mu)u; L^2(\mathbb{R} \times \Omega)\| \\ &\quad - c \|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \Delta_\Omega + (1 + \lambda)^{-2}(\partial_x + \beta)^2)u; L^2(\mathbb{R} \times \Omega)\| \\ &\leq c \|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu)u; L^2(\mathbb{R} \times \Omega)\|. \end{aligned}$$

If T is sufficiently large, then $\epsilon c < 1$. This together with (8.7) gives

$$\|\chi_T u; H_0^2(\mathcal{G})\| \leq C \|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) \chi_T u; L^2(\mathcal{G})\|, \quad (8.10)$$

where the constant $C = c(1 - \epsilon c)^{-1} c_2 c_3$ is independent of $u \in C_0^\infty(\bar{\mathcal{G}})$. By continuity the estimate (8.10) extends to all $u \in H_0^2(\mathcal{G})$.

Now we combine (8.10) with (8.5), and arrive at the estimates

$$\begin{aligned} \|u; H_0^2(\mathcal{G})\| &\leq \|\chi_T u; H_0^2(\mathcal{G})\| + \|\rho_T u; H_0^2(\mathcal{G})\| \\ &\leq C(\|\chi_T (e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) u; L^2(\mathcal{G})\| + \| [e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}, \chi_T] u; L^2(\mathcal{G}) \| \\ &\quad + \|\rho_T (e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) u; L^2(\mathcal{G})\| + \|\rho_T u; L^2(\mathcal{G})\|) \\ &\leq C(\|(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) u; L^2(\mathcal{G})\| + \|\rho_T u; L^2(\mathcal{G})\|). \end{aligned} \quad (8.11)$$

Here we used that $\rho_T = 1$ on the support of the commutator $[e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}, \chi_T]$, and hence

$$\| [e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}, \chi_T] u; L^2(\mathcal{G}) \| \leq C \|\rho_T u; H_0^2(\mathcal{G})\|.$$

For an arbitrary positive weight w we have $\|\rho_T u; L^2(\mathcal{G})\| \leq C \|w u; L^2(\mathcal{G})\|$ with an independent of $u \in H_0^2(\mathcal{G})$ constant C . Thus the estimate (8.3) is a direct consequence of (8.11). By the Peetre's lemma we conclude that the range of the operator (8.1) is closed and the kernel is finite-dimensional.

Clearly, the graph norm $\|u; L^2(\mathcal{G})\| + \|e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} u\|$ of $u \in C_0^\infty(\bar{\mathcal{G}})$ is majorized by $\|u; H_0^2(\mathcal{G})\|$. The estimate (8.3) with $w \equiv 1$ implies that the norm $\|u; H_0^2(\mathcal{G})\|$ is majorized by the graph norm of u . Since the set $C_0^\infty(\bar{\mathcal{G}})$ is dense in $D(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})$ and in $H_0^2(\mathcal{G})$, this proves assertion 1.

In order to see that the cokernel $\text{coker}(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu) = \ker((e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})^* - \bar{\mu})$ of the operator (8.1) is finite-dimensional (if μ , λ , and β does not meet the condition (8.2)) we derive the coercive estimate

$$\|u; H_0^2(\mathcal{G})\| \leq C(\|((e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})^* - \bar{\mu})u; L^2(\mathcal{G})\| + \|wu; L^2(\mathcal{G})\|) \quad (8.12)$$

for the adjoint $(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})^*$ of the m -sectorial operator $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s}$ and apply the Peetre's lemma. The m -sectorial operator $(e^{\beta s} \Delta_{\lambda, r} e^{-\beta s})^*$ corresponds to the closed densely defined sectorial form $\overline{q_{\lambda, r}[u, u]}$ with the domain $\mathring{H}^1(\mathcal{G})$. The proof of the estimate (8.12) is similar to the proof of (8.4), we omit it.

We have proved that the operator (8.1) is Fredholm provided the condition (8.2) is not satisfied. Now we assume that the condition (8.2) is met, and show that the operator (8.1) is not Fredholm.

Let χ be a smooth cutoff function on the real line, such that $\chi(x) = 1$ for $|x-3| \leq 1$ and $\chi(x) = 0$ for $|x-3| \geq 2$. Consider the functions

$$u_\ell(x, y) = \chi(x/\ell) \exp(i(1+\lambda)\sqrt{\mu-\nu}x - \beta x) \Phi(y), \quad (x, y) \in \mathbb{R} \times \Omega, \quad (8.13)$$

where Φ is an eigenfunction of Δ_Ω , corresponding to the eigenvalue $\nu \in \sigma(\Delta_\Omega)$. The exponent in (8.13) is an oscillating function of x . Straightforward calculation shows that

$$\|(\Delta_\Omega - (1+\lambda)^{-2}(\partial_x + \beta)^2 - \mu)u_\ell; L^2(\mathbb{R} \times \Omega)\| \leq C, \quad \|u_\ell; H_0^2(\mathbb{R} \times \Omega)\| \rightarrow \infty \quad (8.14)$$

as $\ell \rightarrow +\infty$. Similarly to (8.8) we conclude that

$$\|((e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \Delta_\Omega + (1+\lambda)^{-2}(\partial_x + \beta)^2)u_\ell; L^2(\mathbb{R} \times \Omega)\| \leq \epsilon_\ell \|u_\ell; H_0^2(\mathbb{R} \times \Omega)\|, \quad (8.15)$$

where $\epsilon_\ell \rightarrow 0$ as $\ell \rightarrow +\infty$. Let the functions $u_\ell = u_\ell \circ \kappa^{-1}$ be extended from \mathcal{C} to \mathcal{G} by zero. If, on the contrary, the operator (8.1) is Fredholm, then by the Peetre's lemma the estimate (8.4) holds with any weight w , such that $H_0^2(\mathcal{G}) \hookrightarrow L^2(\mathcal{G}; w)$ is a compact embedding. Without loss of generality we can assume that $\|wu_\ell; L^2(\mathcal{G})\| \leq C$ for all $\ell \geq 1$. After the change of variables $(\zeta, \eta) \mapsto (x, y)$ the estimate (8.4) implies

$$\|u_\ell; H_0^2(\mathbb{R} \times \Omega)\| \leq C(\|((e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} - \mu)u_\ell; L^2(\mathbb{R} \times \Omega)\| + 1),$$

where the function $e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} u_\ell = (e^{\beta s} \Delta_{\lambda, r} e^{-\beta s} u_\ell) \circ \kappa$ is extended from $\mathbb{R}_+ \times \overline{\Omega}$ to $\mathbb{R} \times \overline{\Omega}$ by zero. This together with (8.15) justifies the estimate

$$\|u_\ell; H_0^2(\mathbb{R} \times \Omega)\| \leq C(\|(\Delta_\Omega - (1+\lambda)^{-2}(\partial_x + \beta)^2 - \mu)u_\ell; L^2(\mathbb{R} \times \Omega)\| + 1),$$

which contradicts (8.14). \square

9. Problem with finite PMLs. Consider the truncated domain \mathcal{G}_R with piecewise smooth boundary, see (2.9). Introduce the Sobolev space $H_0^2(\mathcal{G}_R)$ as the completion of the set $C_0^\infty(\overline{\mathcal{G}_R})$ in the norm $\|v; H_0^2(\mathcal{G}_R)\| = (\sum_{\ell+|m|\leq 2} \|\partial_\zeta^\ell \partial_\eta^m v; L^2(\mathcal{G}_R)\|^2)^{1/2}$. In this section we study the problem with finite PMLs: *Given $g \in L^2(\mathcal{G}_R)$ find a solution $v \in H_0^2(\mathcal{G}_R)$ of the equation*

$$(\Delta_{\lambda, r} - \mu_0)v = g \text{ in } \mathcal{G}_R, \quad (9.1)$$

where $R > r$. The next theorem presents a stability result for this problem.

THEOREM 9.1. *Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is not an eigenvalue of the selfadjoint Dirichlet Laplacian Δ in $L^2(\mathcal{G})$. Take a sufficiently large $r > 0$ and $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$. Then there exists $R_0 > r$ such that for all $R > R_0$ and $g \in L^2(\mathcal{G}_R)$ the equation (9.1) has a unique solution $v \in H_0^2(\mathcal{G}_R)$. Moreover, the estimate*

$$\|v; H_0^2(\mathcal{G}_R)\| \leq C \|g; L^2(\mathcal{G}_R)\| \quad (9.2)$$

holds with an independent of $R > R_0$ and g constant C .

The proof of Theorem 9.1 will be carried out by the compound expansion technique. This requires construction of an approximate solution to the equation (9.1) compounded of solutions to limit problems. As the first limit problem we take the problem with infinite PMLs. As the second limit problem we take a Dirichlet problem in the semi-cylinder $(-\infty, R) \times \Omega$ studied in the next lemma.

LEMMA 9.2. *Introduce the weighted Sobolev space $H_{0,\beta}^2((-\infty, R) \times \Omega)$ of functions satisfying the homogeneous Dirichlet boundary condition as the completion of the set $C_0^\infty((-\infty, R) \times \overline{\Omega})$ with respect to the norm*

$$\|u; H_{0,\beta}^2((-\infty, R) \times \Omega)\| = \left(\sum_{\ell+|m| \leq 2} \int_{-\infty}^R \|e^{-\beta x} \partial_x^\ell \partial_y^m u(x); L^2(\Omega)\|^2 dx \right)^{1/2}.$$

Let $L_\beta^2((-\infty, R) \times \Omega)$ be the weighted L^2 -space with the norm

$$\|f; L_\beta^2((-\infty, R) \times \Omega)\| = \left(\int_{-\infty}^R \|e^{-\beta x} f(x); L^2(\Omega)\|^2 dx \right)^{1/2}.$$

Assume that $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$, $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$, and β is in the interval (7.1). Then for any $f \in L_\beta^2((-\infty, R) \times \Omega)$ there exists a unique solution $u \in H_{0,0}^2((-\infty, R) \times \Omega)$ to the equation

$$(\Delta_\Omega - (1 + \lambda)^{-2} \partial_x^2 - \mu_0)u = f. \quad (9.3)$$

Moreover, the estimate

$$\|u; H_{0,\beta}^2((-\infty, R) \times \Omega)\| \leq C \|f; L_\beta^2((-\infty, R) \times \Omega)\| \quad (9.4)$$

holds, where the constant C is independent of f and R .

Proof. It suffices to prove the assertion for $R = 0$. Then the general case can be obtained by the change of variables $x \mapsto x - R$.

The set $C_c^\infty((-\infty, 0) \times \Omega)$ of smooth functions with compact supports is dense in $L_\beta^2((-\infty, 0) \times \Omega)$. We first assume that $f \in C_c^\infty((-\infty, 0) \times \Omega)$ and extend f to a function in $C_c^\infty(\mathbb{R} \times \Omega)$ by setting $f(-x) = -f(x)$ for $x < 0$. Consider the equation (9.3) in the infinite cylinder $\mathbb{R} \times \Omega$. As is known [36], the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x) dx$ is an entire function of ξ with values in $L^2(\Omega)$; it is rapidly decaying at infinity in any strip $\{\xi \in \mathbb{C} : |\Im \xi| < \beta\}$ in the sense that the estimates $\|\hat{f}(\xi); L^2(\Omega)\| \leq C_{\beta,k} (1 + |\xi|)^{-k}$ hold for $k = 0, 1, 2, \dots$ and some constants $C_{\beta,k}$. Since β is in the interval (7.1), the distance d between the set $\{\mu_0 - (1 + \lambda)^{-2} \xi^2 : 0 \leq \Im \xi < \beta\}$ and the spectrum $\sigma(\Delta_\Omega)$ of the selfadjoint operator Δ_Ω in $L^2(\Omega)$ with domain $H_0^2(\Omega)$ is positive. Hence for

$$\Psi(\xi) = (\Delta_\Omega + (1 + \lambda)^{-2} \xi^2 - \mu_0)^{-1} \hat{f}(\xi), \quad 0 \leq \Im \xi < \beta,$$

we have the estimate $\|\Psi(\xi); L^2(\Omega)\|^2 \leq d^{-2} \|\hat{f}(\xi); L^2(\Omega)\|^2$. This together with the elliptic coercive estimate

$$\|\Psi(\xi); H_0^2(\Omega)\|^2 \leq c(\|\hat{f}(\xi); L^2(\Omega)\|^2 + \|\Psi(\xi); L^2(\Omega)\|^2)$$

for the Dirichlet Laplacian Δ_Ω gives

$$\|\Psi(\xi); H_0^2(\Omega)\|^2 \leq (c + d^{-2}) \|\hat{f}(\xi); L^2(\Omega)\|^2, \quad 0 \leq \Im \xi < \beta. \quad (9.5)$$

The differential operator $\Delta_\Omega - (1 + \lambda)^{-2} \partial_x^2$ is strongly elliptic. Therefore the local coercive estimate

$$\|\varrho u; H_0^2(\mathbb{R} \times \Omega)\|^2 \leq c \left(\|\varsigma f; L^2(\mathbb{R} \times \Omega)\|^2 + \|\varsigma u; L^2(\mathbb{R} \times \Omega)\|^2 \right) \quad (9.6)$$

is valid, where ϱ and ς are smooth functions of the variable x with compact supports such that $\varrho \not\equiv 0$ and $\varrho \varsigma = \varrho$. We substitute $u(x, y) = e^{i\xi x} \Psi(\xi, y)$ into (9.6). After simple manipulations we arrive at the estimate

$$\sum_{\ell+|m|\leq 2} |\xi|^{2\ell} \|\partial_y^m \Psi(\xi); L^2(\Omega)\|^2 \leq C \left(\|\hat{f}(\xi); L^2(\Omega)\|^2 + \|\Psi(\xi); L^2(\Omega)\|^2 \right), \quad (9.7)$$

where the constant C depends on ϱ and ς , but not on ξ or f . If $|\xi| > C$ with sufficiently large $C > 0$, then the last term in (9.7) can be neglected. This together with (9.5) justifies the estimate

$$\sum_{\ell+|m|\leq 2} |\xi|^{2\ell} \|\partial_y^m \Psi(\xi); L^2(\Omega)\|^2 \leq C \|\hat{f}(\xi); L^2(\Omega)\|^2, \quad (9.8)$$

where $0 \leq \Im \xi < \beta$ and the constant C is independent of ξ and $\Psi(\xi)$. Therefore the analytic in strip $0 \leq \Im \xi < \beta$ function $\xi \mapsto \Psi(\xi) \in H_0^2(\Omega)$ is rapidly decaying at infinity. This together with the Cauchy integral theorem allows us to replace the contour of integration in the inverse Fourier transformation $u(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\xi x} \Psi(\xi) d\xi$. We obtain

$$\begin{aligned} u(x) &= (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\xi x} (\Delta_\Omega + (1 + \lambda)^{-2} \xi^2 - \mu_0)^{-1} \hat{f}(\xi) d\xi \\ &= (2\pi)^{-1} \int_{\xi - i\beta \in \mathbb{R}} e^{-i\xi x} (\Delta_\Omega + (1 + \lambda)^{-2} \xi^2 - \mu_0)^{-1} \hat{f}(\xi) d\xi. \end{aligned}$$

The Parseval equality gives

$$2\pi \int_{\mathbb{R}} \|e^{-\beta x} \partial_x^\ell \partial_y^m u(x); L^2(\Omega)\|^2 dx = \int_{\xi - i\beta \in \mathbb{R}} |\xi|^{2\ell} \|\partial_y^m \Psi(\xi); L^2(\Omega)\|^2 d\xi,$$

$$2\pi \int_{\mathbb{R}} \|e^{-\beta x} f(x); L^2(\Omega)\|^2 dx = \int_{\xi - i\beta \in \mathbb{R}} \|\hat{f}(\xi); L^2(\Omega)\|^2 d\xi.$$

Integrating (9.8) with respect to ξ , $\xi - i\beta \in \mathbb{R}$, we deduce the estimate

$$\sum_{\ell+|m|\leq 2} \int_{-\infty}^{\infty} \|e^{-\beta x} \partial_x^\ell \partial_y^m u(x); L^2(\Omega)\|^2 dx \leq C \int_{-\infty}^{\infty} \|e^{-\beta x} f(x); L^2(\Omega)\|^2 dx; \quad (9.9)$$

in the case $\beta = 0$ this estimate takes the form $\|u; H_0^2(\mathbb{R} \times \Omega)\| \leq C\|f; L^2(\mathbb{R} \times \Omega)\|$. Thus for any $f \in C_c^\infty(\mathbb{R} \times \Omega)$ there exists a solution $u \in H_0^2(\mathbb{R} \times \Omega)$ to the equation (9.3) and the estimate (9.9) holds with any β in the interval (7.1). Usual argument on smoothness of solutions to elliptic problems gives $u \in C^\infty(\mathbb{R} \times \overline{\Omega})$. From the equality $f(-x) = -f(x)$ it follows that $\hat{f}(-\xi) = -\hat{f}(\xi)$ and therefore $\Psi(\xi) = -\Psi(-\xi)$. Hence $u(x) = -u(-x)$ and $u(0) = 0$. As in the proof of Proposition 4.4 we conclude that in the norm of $H_{0,\beta}^2((0, \infty) \times \Omega)$ one can approximate u by functions in $C_0^\infty((0, \infty) \times \Omega)$. Hence $u \in H_{0,\beta}^2((-\infty, 0) \times \Omega)$. By continuity our construction extends to all $f \in L_\beta^2((-\infty, 0) \times \Omega)$. In particular, for any $f \in L_0^2((-\infty, 0) \times \Omega)$ we can find a solution $u \in H_{0,0}^2((-\infty, 0) \times \Omega)$ to the equation (9.3). If $f \in L_\beta^2((-\infty, 0) \times \Omega)$ with some β in the interval (7.1), then the estimate (9.4) with $R = 0$ is a direct consequence of (9.9). It remains to note that $u \in H_{0,0}^2((-\infty, 0) \times \Omega)$ is a unique solution as our argument also shows that for any $f \in L_0^2((-\infty, 0) \times \Omega)$ the adjoint equation $(\Delta_\Omega - (1 + \bar{\lambda})^{-2}\partial_x^2 - \mu_0)u = f$ is solvable in the space $H_{0,0}^2((-\infty, 0) \times \Omega)$. \square

Now we are in position to prove Theorem 9.1.

Proof. The scheme of the proof is similar to the one we used in the proof of [20, Theorem 4.1]. We rely on a modification of the compound expansion method [27]. We say that $w = w(R) \in H_0^2(\mathcal{G}_R)$ is an approximate solution of the problem with finite PML if the following conditions are satisfied:

- i. The estimate $\|w; H_0^2(\mathcal{G}_R)\| \leq c\|g; L^2(\mathcal{G}_R)\|$ holds with an independent of g and R constant c ;
- ii. The estimate $\|(\Delta_{\lambda,r} - \mu_0)w - g; L^2(\mathcal{G}_R)\| \leq C_R\|g; L^2(\mathcal{G}_R)\|$ is valid, where the constant C_R is independent of g and $C_R \rightarrow 0$ as $R \rightarrow +\infty$.

Due to condition i w continuously depends on g . Condition ii implies that the discrepancy, left by w in the equation (9.1), tends to zero as $R \rightarrow +\infty$. Once an approximate solution w is found, it is not hard to verify the assertion of the theorem.

Let $\rho \in C^\infty(\mathbb{R})$ be a cutoff function such that $\rho(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1/2$. We set $\varrho_R = \rho(x - R)$, $\varrho_R(\zeta, \eta) = \rho_R \circ \varkappa^{-1}(\zeta, \eta)$ for $(\zeta, \eta) \in \overline{\mathcal{C}}$, and $\varrho_R(\zeta, \eta) = 1$ for $(\zeta, \eta) \in \overline{\mathcal{G}} \setminus \overline{\mathcal{C}}$. Let $\mathcal{F} = \varrho_{R/2}g$ and $f = (g - \mathcal{F})|_{\mathcal{G}_R} \circ \varkappa$. We extend \mathcal{F} from \mathcal{G}_R to \mathcal{G} and f from $(0, R) \times \Omega$ to $(-\infty, R) \times \Omega$ by zero. We find an approximate solution w compounded of $u_{\lambda,r} = (\Delta_{\lambda,r} - \mu_0)^{-1}\mathcal{F}$ and a solution $u \in H_{0,0}^2((-\infty, R) \times \Omega)$ to the equation (9.3) in the form

$$w = \varrho_R u_{\lambda,r} + (1 - \varrho_{R/3})(u \circ \varkappa^{-1});$$

here the second term in the right hand side is extended from $\mathcal{G}_R \cap \mathcal{C}$ to \mathcal{G}_R by zero.

Let us show that w is an approximate solution. Observe that on the support of f we have $e^{\beta s} \leq Ce^{\beta R/2}$ and on the support of f we have $e^{-\beta x} \leq Ce^{-\beta R/2}$ uniformly in R . Hence

$$\|e^{\beta s}\mathcal{F}; L^2(\mathcal{G})\| + e^{\beta R}\|f; L_\beta^2((-\infty, R) \times \Omega)\| \leq Ce^{\beta R/2}\|g; L^2(\mathcal{G})\| \quad (9.10)$$

with an independent of R and g constant C . Similarly to (8.7) we conclude that

$$\begin{aligned} \|w; H_0^2(\mathcal{G}_R)\|^2 &\leq \|\varrho_R u_{\lambda,r}; H_0^2(\mathcal{G})\|^2 + c\left(\|\Delta_{\lambda,r}(1 - \varrho_{R/3})u; L^2((-\infty, R) \times \Omega)\| \right. \\ &\quad \left. + \|u; L^2((-\infty, R) \times \Omega)\| \right)^2 \leq C\|u_{\lambda,r}; H_0^2(\mathcal{G})\|^2 + C\|u; H_{0,0}^2((-\infty, 0) \times \Omega)\|^2, \end{aligned}$$

where C is independent of R and $\Delta_{\lambda,r}$ is the operator (8.6). This together with the estimates (9.4), (9.10) for $\beta = 0$ and Theorem 6.1.1 implies that the condition i is satisfied.

Let us verify the condition ii. We have

$$\begin{aligned} (\Delta_{\lambda,r} - \mu_0)w - g &= [\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r} + (1 + \lambda)^{-2}([\partial_x^2, \rho_{R/3}]u) \circ \varkappa^{-1} \\ &\quad + \left((\Delta_{\lambda,r} - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2)(1 - \rho_{R/3})u \right) \circ \varkappa^{-1}. \end{aligned} \quad (9.11)$$

The support of the term $[\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r}$ is a subset of the image of $(R, R + 1/2) \times \Omega$ under the diffeomorphism \varkappa . On this support the weight $e^{\beta s}$ is bounded from below by $ce^{\beta R}$ uniformly in $R > 0$. As a consequence we get the uniform in R estimates

$$\|[\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r}; L^2(\mathcal{G}_R)\| \leq C_1 e^{-\beta R} \|e^{\beta s}u_{\lambda,r}; H_0^2(\mathcal{G})\| \leq C_2 e^{-\beta R} \|e^{\beta s}\mathcal{F}; L^2(\mathcal{G})\|, \quad (9.12)$$

where we used Theorem 7.1. Now we estimate the second term in the right hand side of (9.11). On the support of $[\partial_x^2, \rho_{R/3}]u$ we have $e^{-\beta x} \geq Ce^{-\beta R/3}$. Relying on (9.4) we obtain

$$\begin{aligned} \|(1 + \lambda)^{-2}([\partial_x^2, \rho_{R/3}]u) \circ \varkappa^{-1}; L^2(\mathcal{G}_R)\| &\leq c \|[\partial_x^2, \rho_{R/3}]u; L^2((-\infty, R) \times \Omega)\| \\ &\leq C_1 e^{\beta R/3} \|u; H_{0,\beta}^2((-\infty, R) \times \Omega)\| \leq C_2 e^{\beta R/3} \|\mathcal{F}; L^2((-\infty, R) \times \Omega)\|. \end{aligned} \quad (9.13)$$

Finally, consider the last term in the right hand side of (9.11). On the support of $(1 - \rho_{R/3})u$ the coefficients of the operator $\Delta_{\lambda,r} - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2$ tend to zero as $R \rightarrow +\infty$, cf. (3.4) and (8.6). This together with the estimate (9.4) for $\beta = 0$ gives

$$\begin{aligned} \left\| \left((\Delta_{\lambda,r} - \Delta_\Omega + (1 + \lambda)^{-2}\partial_x^2)(1 - \rho_{R/3})u \right) \circ \varkappa^{-1}; L^2(\mathcal{G}_R) \right\| \\ \leq c_R \|\mathcal{F}; L_0^2((-\infty, R) \times \Omega)\|, \end{aligned} \quad (9.14)$$

where $c_R \rightarrow 0$ as $R \rightarrow +\infty$. From (9.10)–(9.14) it follows that w meets the condition ii. Thus $w = w(R)$ is indeed an approximate solution to the problem with finite PML.

Now we are in position to prove the assertion of the theorem. Observe that $(\Delta_{\lambda,r} - \mu_0)w - g = \mathcal{O}(R)g$ with some operator $\mathcal{O}(R)$ in $L^2(\mathcal{G}_R)$, whose norm $\|\mathcal{O}(R)\|$ tends to zero as $R \rightarrow +\infty$ because of the condition ii on w . For all $R > R_0$ with a sufficiently large R_0 we have $\|\mathcal{O}(R)\| \leq \|\mathcal{O}(R_0)\| < 1$. Hence there exists the inverse $(I + \mathcal{O}(R))^{-1} : L^2(\mathcal{G}_R) \rightarrow L^2(\mathcal{G}_R)$ and its norm is bounded by the constant $1/(1 - \|\mathcal{O}(R_0)\|)$ uniformly in $R > R_0$. We set $\tilde{g} = (I + \mathcal{O}(R))^{-1}g$. In the same way as before we construct the approximate solution w for the problem (9.1), where g is replaced by \tilde{g} . Then for $v = w$ we have $(\Delta_{\lambda,r} - \mu_0)v = \tilde{g} + \mathcal{O}(R)\tilde{g} = g$ and

$$\|v; H_0^2(\mathcal{G}_R)\| \leq c \|\tilde{g}; L^2(\mathcal{G}_R)\| \leq c/(1 - \|\mathcal{O}(R_0)\|) \|g; L^2(\mathcal{G}_R)\|,$$

where C is independent of $R > R_0$. Thus for $R > R_0$ and $g \in L^2(\mathcal{G}_R)$ there exists a solution $v \in H_0^2(\mathcal{G}_R)$ to the equation (9.1) satisfying the estimate (9.2), where the constant C is independent of R and g . In the remaining part of the proof we show that this solution is unique.

Let $\Delta_{\lambda,r}^R$ be the unbounded operator in $L^2(\mathcal{G}_R)$ such that for any v in its domain $H_0^2(\mathcal{G}_R)$ we have $\Delta_{\lambda,r}^R v = \Delta_{\lambda,r} v$. Note that $\Delta_{\lambda,r}^R$ is the operator of the problem with finite PML. To the operator $\Delta_{\lambda,r}^R$ there corresponds the quadratic form

$$q_{\lambda,r}^R[v, v] = \int_{\mathcal{G}_R} \left\langle \left(\det \mathbf{e}_{\lambda,r} \right)^{1/2} \mathbf{e}_{\lambda,r}^{-1} \nabla_{\zeta\eta} v, \nabla_{\zeta\eta} \left(\det \mathbf{e}_{\lambda,r} \right)^{-1/2} v \right\rangle d\zeta d\eta$$

in $L^2(\mathcal{G}_R)$ with the domain $H_0^2(\mathcal{G}_R)$. In the same way as in Section 4 we conclude that the form $q_{\lambda,r}^R$ admits a densely defined sectorial closure with the domain $\mathring{H}^1(\mathcal{G}_R)$, where $\mathring{H}^1(\mathcal{G}_R)$ is the completion of $H_0^2(\mathcal{G}_R)$ with respect to the norm

$\|v; \mathring{H}^1(\mathcal{G}_R)\| = (\sum_{\ell+|m| \leq 1} \|\partial_\zeta^\ell \partial_\eta^m v; L^2(\mathcal{G}_R)\|^2)^{1/2}$. This together with the argument above implies that all $\mu < 0$ with sufficiently large absolute value are regular points of the operator $\Delta_{\lambda,r}^R$. Hence the sectorial operator $\Delta_{\lambda,r}^R$ coincides with its m-sectorial Friedrichs extension. Moreover, thanks to the argument above we know that under the assumptions of theorem for any $g \in L^2(\mathcal{G}_R)$ there exists $v \in H_0^2(\mathcal{G}_R)$ such that $(\Delta_{\lambda,r}^R - \mu_0)v = g$. Similarly, one can study the adjoint m-sectorial operator $(\Delta_{\lambda,r}^R)^*$. It turns out that under the assumptions of theorem for any $g \in L^2(\mathcal{G}_R)$ there exists v in the domain $H_0^2(\mathcal{G}_R)$ of $(\Delta_{\lambda,r}^R)^*$ such that $((\Delta_{\lambda,r}^R)^* - \mu_0)v = g$. Therefore the equation (9.1) is uniquely solvable in $H_0^2(\mathcal{G}_R)$. \square

In the next theorem we show that under some natural assumptions solutions $v = v(R)$ of the problem with finite PMLs converge in the domain \mathcal{G}_r to outgoing or incoming solutions u_\pm with an exponential rate as $R \rightarrow +\infty$. In other words, we estimate the error produced by truncation of infinite PMLs. Solutions to the problem with finite PMLs can be found numerically with the help of finite element solvers; certainly, discretization produces yet another error that we do not estimate here.

THEOREM 9.3. *Assume that $\mu_0 \in \mathbb{R} \setminus \sigma(\Delta_\Omega)$ is not an eigenvalue of the selfadjoint Dirichlet Laplacian Δ in $L^2(\mathcal{G})$, the parameter $r > 0$ is sufficiently large in the sense of Remark 4.2, $\lambda \in \mathcal{D}_\alpha \setminus \mathbb{R}$, and β is in the interval (7.1). Let $f \in \mathcal{H}_\alpha(\mathcal{G})$ satisfy the inclusion $e^{\beta s}(f \circ \vartheta_{\lambda,r}) \in L^2(\mathcal{G})$, where s is the same function as in Theorem 7.1. In (9.1) we set $g = f \circ \vartheta_{\lambda,r}$. Then there exists $R_0 > r$ such that for $R > R_0$ a unique solution $v = v(R) \in H_0^2(\mathcal{G}_R)$ of the problem with finite PMLs converges in \mathcal{G}_r .*

1. to the outgoing solution $u_- \in H_{0,\text{loc}}^2(\mathcal{G})$ of the equation $(\Delta - \mu_0)u = f$ in the case $\Im \lambda > 0$
2. to the incoming solution $u_+ \in H_{0,\text{loc}}^2(\mathcal{G})$ of the equation $(\Delta - \mu_0)u = f$ in the case $\Im \lambda < 0$

in the sense that as $R \rightarrow +\infty$ the estimate

$$\sum_{\ell+|m| \leq 2} \|\partial_\zeta^\ell \partial_\eta^m (u_\pm - v_R); L^2(\mathcal{G}_r)\|^2 \leq C e^{-2\beta R} \|e^{\beta s}(f \circ \vartheta_{\lambda,r}); L^2(\mathcal{G})\|^2 \quad (9.15)$$

holds with a constant C independent of $R > R_0$ and f .

Let us remark here that the assumptions of Theorem 9.3 on the right hand side f are a priori met for all $f \in L^2(\mathcal{G})$ such that $f \upharpoonright \mathcal{C} = F \circ \varkappa$ with some $F \in \mathcal{E}$; here \mathcal{E} is the algebra defined in Section 5 and \varkappa is the diffeomorphism (2.1). From Lemma 5.2 it follows that the set of functions f satisfying the assumptions of Theorem 9.3 is dense in $L^2(\mathcal{G})$. In particular, we can take any $f \in L^2(\mathcal{G})$ supported in \mathcal{G}_r .

Proof. Let $\rho \in C^\infty(\mathbb{R})$ be a cutoff function such that $\rho(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1/2$. We set $\varrho_R = \rho(x - R)$, $\varrho_R(\zeta, \eta) = \rho_R \circ \varkappa^{-1}(\zeta, \eta)$ for $(\zeta, \eta) \in \overline{\mathcal{C}}$, and $\varrho_R(\zeta, \eta) = 1$ for $(\zeta, \eta) \in \overline{\mathcal{G}} \setminus \overline{\mathcal{C}}$. Thanks to Theorem 6.1.3 it suffices to prove the estimate (9.15) with u_\pm replaced by $\varrho_R u_{\lambda,r}$. The difference $\varrho_R u_{\lambda,r} - v_R \in H_0^2(\mathcal{G}_R)$ satisfies the problem (9.1) with $g = (\varrho_R - 1)(f \circ \vartheta_{\lambda,r}) + [\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r}$. Observe that

$$\begin{aligned} \|(\varrho_R - 1)(f \circ \vartheta_{\lambda,r}); L^2(\mathcal{G}_R)\| &\leq C e^{-\beta R} \|e^{\beta s}(f \circ \vartheta_{\lambda,r}); L^2(\mathcal{G})\|, \\ \|[\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r}; L^2(\mathcal{G}_R)\| &\leq C e^{-\beta R} \|e^{\beta s}u_{\lambda,r}; H_0^2(\mathcal{G})\|, \end{aligned}$$

because the functions $(\varrho_R - 1)(f \circ \vartheta_{\lambda,r})$ and $[\Delta_{\lambda,r}, \varrho_R]u_{\lambda,r}$, being written in the coordinates (x, y) , are equal to zero for $x < R$, while $e^{\beta s(\zeta, \eta)} = ce^{\beta x}$ for $x \geq R > C$, cf. (2.4). This together with Theorem 7.1 gives

$$\|g; L^2(\mathcal{G}_R)\| \leq c e^{-\beta R} \|e^{\beta s}(f \circ \vartheta_{\lambda,r}); L^2(\mathcal{G})\|. \quad (9.16)$$

By Theorem 9.1 we have

$$\|\varrho_R u_{\lambda,r} - v_R; H_0^2(\mathcal{G}_R)\| \leq C \|g; L^2(\mathcal{G}_R)\|, \quad R > R_0. \quad (9.17)$$

It remains to note that

$$\sum_{\ell+|m|\leq 2} \|\partial_\zeta^\ell \partial_\eta^m (u_\pm - v_R); L^2(\mathcal{G}_r)\|^2 \leq \|\varrho_R u_{\lambda,r} - v_R; H_0^2(\mathcal{G}_R)\|^2, \quad R > R_0 > r.$$

This together with (9.17) and (9.16) completes the proof of the estimate (9.15). \square

REMARK 9.4. *By Theorem 9.3 the rate of convergence of the PML method depends only on the spectral parameter μ_0 and the infinitely distant cross-section Ω . In the case $\mathcal{C} = (0, \infty) \times \Omega$ the quasi-cylindrical domain \mathcal{G} corresponds to a resonator with attached tubular waveguide. In this particular case the results of Theorem 9.3 for $f \in L^2(\mathcal{G})$ with $\text{supp } f \subset \mathcal{G}_r$ can equivalently be obtained by the modal expansions technique, e.g. [5, 6, 7].*

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